$$F_{xy} = \frac{d^2 f}{dy\,dx}$$

$$f(x,y) = 2x^3 + 3xy^2$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$\frac{d^2 f}{dx^2} = 12x$$

$$\frac{d^3 f}{dx^3} = 12$$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\frac{df}{dy}$$

$$\frac{df}{dy} = 6xy$$

$$6xy\frac{d}{dx} = 6y$$

$$\frac{df}{dx} = 6xy$$

$$\frac{d^2 f}{dx^2} = 6x$$

$$\frac{d^3 f}{dy^3} = 0$$

$$\frac{d^2 f}{dy\,dx} = \frac{d}{dy}\frac{df}{dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$6x^2 + 3y^2\frac{d}{dy} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$u=(x^2+y^2+z^2)^{-\frac{1}{2}} \Rightarrow \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$$

$$\frac{du}{dx}=2x\cdot\frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dx^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-x\cdot 2x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dy}=2y\cdot\frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-y(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dy^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-y\cdot 2y\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dz}=2z\cdot-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-z\cdot(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dz^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-z\cdot 2z\cdot\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \left.\frac{d^2 z}{d_x d y}\right|_{(1,1)} = ? \qquad \frac{d}{dx}\left(\frac{dz}{dy}\right) = z_{yx}$$

$$\frac{dz}{dy} = x^2\left(\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) = \frac{x.x^2}{x^2+y^2} = \frac{x^3}{x^2+y^2}$$

$$\frac{d}{dx}\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2)-2x^4}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$\left.\frac{d^2 z}{dx dy}\right|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y

$f(x,y)$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$\Delta x \to 0 \quad \Delta y \to 0$

$\Delta x \cong dx \quad \Delta y \cong dy$

$$\Delta z = \frac{df}{dx}\cdot\Delta x + \frac{df}{dy}\cdot\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$\Delta x \to 0 \quad \Delta y \to 0$

$$\Delta z = \underbrace{\frac{df}{dx}\,dx + \frac{df}{dy}\,dy}_{\text{Important Part = Diffentiation of } f} + \varepsilon_1 dx + \varepsilon_2 dy$$

$$z = f(x,y) = x^2 y - 3y \Rightarrow \Delta z = ? \quad dz = ?$$

$$\Delta x = -0{,}01 \qquad \Delta y = 0{,}02$$

$$\Delta z = \left[(x+\Delta x)^2 (y+\Delta y) - 3(y+\Delta y)\right] - (x^2 y - 3y)$$

$$\left[\left(x^2 + 2x\Delta x + (\Delta x)^2\right)(y+\Delta y) - 3(y+\Delta y)\right] - y(x^2 - 3)$$

$$y+\Delta y \left(x^2 + 2x\Delta x + (\Delta x)^2 - 3\right)$$

$$\cancel{yx^2} + y2x\Delta x + y(\Delta x)^2 \cancel{-3y} + \Delta y x^2 + \Delta y 2x\Delta x + \Delta y(\Delta x)^2 - 3\Delta y \cancel{-yx^2} \cancel{+3y}$$

$$\underbrace{2xy\Delta x + x^2\Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x\Delta y + (\Delta x^2)y + \Delta x^2\Delta y}_{\text{ignored}}$$

$$\Delta z = 2xy\Delta x + (x^2 - 3)\Delta y$$

$$\cong$$

$$dz = 2xy\,dx + x^2 - 3\,dy$$

As we seen here when Δx and Δy go to 0 then $dz \cong \Delta z$

$v = x^2 e^{y/x} \Rightarrow dv = ?$ $\qquad dv = \frac{dv}{dx} dx + \frac{dv}{dy} dy \ldots\ldots$

$dx = \left(2x e^{y/x} + \frac{y}{x^2} e^{y/x} x^2\right) dx$

$dy = \left(x e^{y/x}\right) dy$

$$dv = e^{y/x}(2x+y)\,dx + e^{y/x} x \, dy$$

Chain Rule

$z = f(x, y) \qquad x = g(r, s) \qquad y = h(r, s)$

$$\begin{tikzcd}[row sep=small]
 & & r \\
 & x \arrow[ur, dash] \arrow[r, dash] & s \\
z \arrow[ur, dash] \arrow[dr, dash] & & \\
 & y \arrow[r, dash] \arrow[dr, dash] & r \\
 & & s
\end{tikzcd}$$

$$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr}$$

$$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}$$

$z = e^x \sin y$, $x = \ln t$, $y = t^2$ $\frac{dz}{dt} = ?$

$\frac{dz}{dx} = e^x \sin y$

$\frac{dz}{dy} = e^x \cos y$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 2t$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

$$e^x \left(\frac{\sin y}{t} + \cos y \, 2t \right)$$

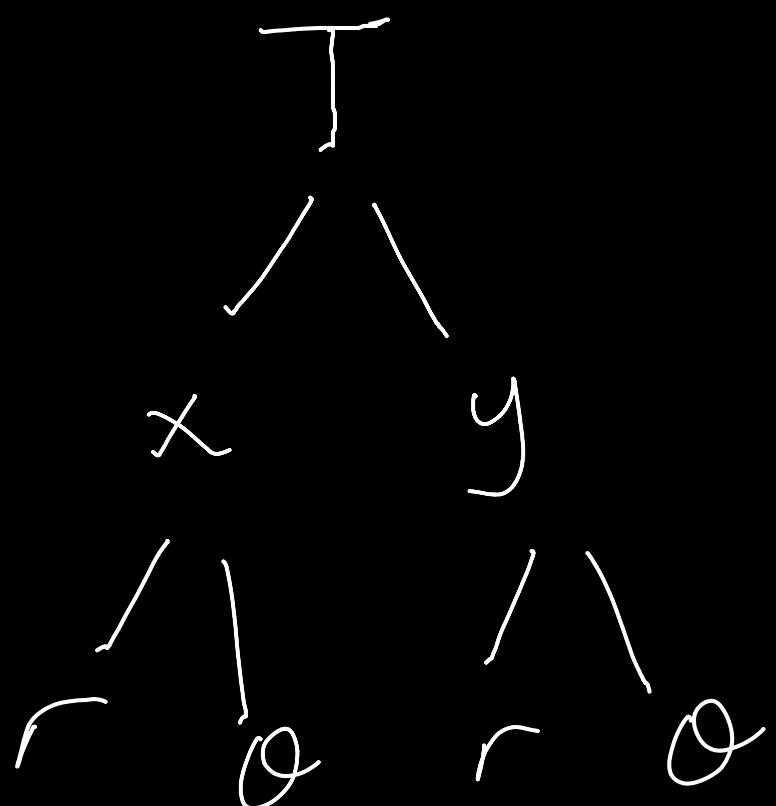
$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) 2t \right.$$

$$\sin(t^2) + 2t^2 \cos(t^2)$$

$$T = x^3 - xy + y^3 \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$\frac{dT}{dr} = ? \qquad \frac{dT}{dr} = \frac{dT}{dx}\cdot\frac{dx}{dr} + \frac{dT}{dy}\cdot\frac{dy}{dr}$$

$$\frac{dT}{d\theta} = ? \qquad \frac{dT}{d\theta} = \frac{dT}{dx}\cdot\frac{dx}{d\theta} + \frac{dT}{dy}\cdot\frac{dy}{d\theta}$$

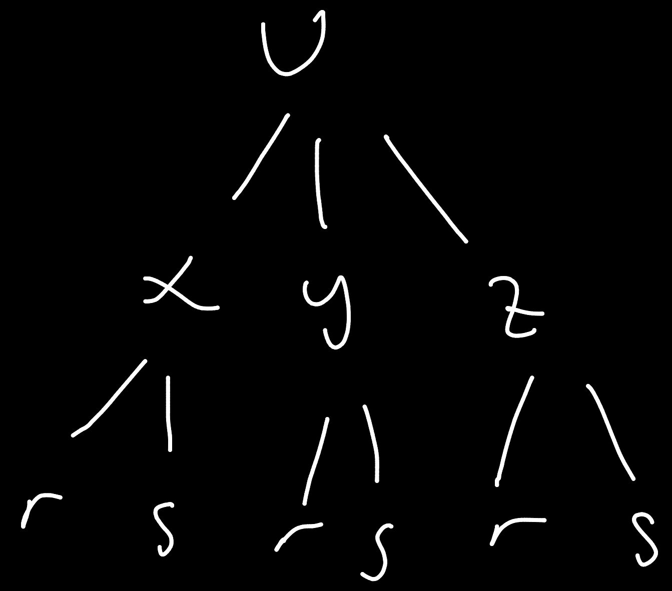


$$\frac{dT}{dr} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{dT}{d\theta} = (3x^2 - y)\cdot -r\sin\theta + (3y^2 - x)\,r\cos\theta$$

$u = z\sin\left(\frac{y}{x}\right)$, $x = 3r^2 + s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$

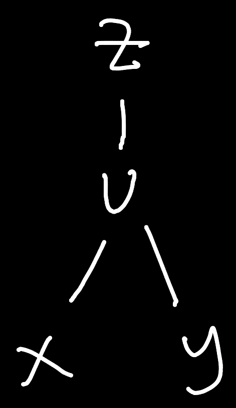
$\frac{du}{ds} = ?$, $\frac{du}{dr} = ?$



$$\frac{du}{ds} = \frac{du}{dx}\cdot\frac{dx}{ds} + \frac{du}{dy}\cdot\frac{dy}{ds} + \frac{du}{dz}\cdot\frac{dz}{ds}$$

$$\frac{du}{dr} = \frac{du}{dx}\cdot\frac{dx}{dr} + \frac{du}{dy}\cdot\frac{dy}{dr} + \frac{du}{dz}\cdot\frac{dz}{dr}$$

$z = f(u) \qquad u = x^2 y \qquad$ prove that $\quad x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{1}{2xy}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial u} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$\frac{1}{2xy}\frac{\partial z}{\partial x} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$$

$x^5+y=t \qquad u=x^3y \qquad \frac{du}{dt}=?$

$x^2+y^3=t^2$

$$\begin{tikzcd}
 & U \arrow[dl, no head] \arrow[dr, no head] & \\
x \arrow[d, no head] & & y \arrow[d, no head] \\
t & & t
\end{tikzcd}$$

$$\frac{du}{dt}=\underbrace{\frac{du}{dx}}_{3x^2y}\cdot\frac{dx}{dt}+\underbrace{\frac{du}{dy}}_{x^3}\cdot\frac{dy}{dt}$$

$$\frac{d}{dt}\left(x^5+y=t\right)=5x^4\frac{dx}{dt}+\frac{dy}{dt}=1$$

$$\frac{d}{dt}\left(x^2+y^3=t^2\right)=2x\frac{dx}{dt}+3y^2\frac{dy}{dt}=2t$$

$$\begin{bmatrix}5x^4 & 1\\ 2x & 3y^2\end{bmatrix}\begin{bmatrix}dx/dt\\ dy/dt\end{bmatrix}\begin{bmatrix}1\\ 2t\end{bmatrix}$$

$$\frac{dx}{dt}=\frac{\begin{vmatrix}1 & 1\\ 2t & 3y^2\end{vmatrix}}{\begin{vmatrix}5x^4 & 1\\ 2x & 3y^2\end{vmatrix}}=\frac{3y^2-2t}{15x^4y^2-2x}$$

$$\frac{dy}{dt}=\frac{\begin{vmatrix}5x^4 & 1\\ 2x & 2t\end{vmatrix}}{\begin{vmatrix}5x^4 & 1\\ 2x & 3y^2\end{vmatrix}}=\frac{10x^4t-2x}{15x^4y^2-2x}$$

$$\frac{du}{dt}=3x^2y\left(\frac{3y^2-2t}{15x^4y^2-2x}\right)+x^3\left(\frac{10x^4t-2x}{15x^4y^2-2x}\right)$$

Jacobian

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$F(u,v,w)=0 \quad G(u,v,w)=0 \quad H(u,v,w)=0$$

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$$F(x,y,u,v)=0 \quad G(x,y,u,v)=0$$

$$\frac{du}{dx} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dx} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{du}{dy} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dy} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$u^2 - v = 3x + y \qquad \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = ?$$

$$u - 2v^2 = x - 2y$$

$$f(u,v,x,y) = 0 \Rightarrow u^2 - v - 3x - y = 0$$

$$g(u,v,x,y) = 0 \Rightarrow u - 2v^2 - x + 2y = 0$$

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{12v - 1}{-8u + 1}$$

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & -1 \\ 2 & -4v \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4v + 2}{-8u + 1}$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{-2u + 3}{-8v + 1}$$

$$\frac{\partial v}{\partial y} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4u + 1}{-8u + 1}$$

$z^3 - xz - y = 0$ $\quad \frac{d^2 z}{dx\,dy} = ?$

It is closed-form
but we have 2 independent but
1 dependent variable that's why
we can't use jacobian

$$\frac{dz}{dx\,dy} = \frac{d}{dx}\left(\frac{dz}{dy}\right)$$

$$\frac{dz}{dy} = 3z^2\frac{dz}{dy} - x\frac{dz}{dy} - 1 = 0 \qquad \frac{dz}{dy} = \frac{1}{3z^2 - x}$$

we still don't know

$$\frac{d}{dx}\left(\frac{1}{3z^2 - x}\right) = \frac{d(3z^2 - x)^{-1}}{dx} = -(3z^2 - x)^{-2} \cdot \left(6z\frac{dz}{dx} - 1\right) = \frac{6z\frac{dz}{dx} - 1}{-(3z^2 - x)^2}$$

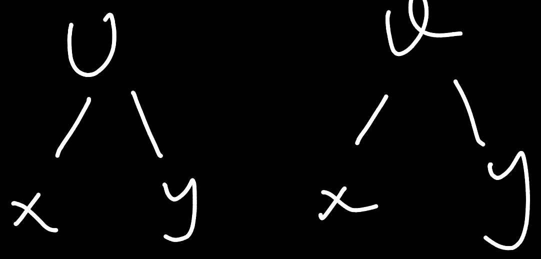
$$\frac{dz}{dx} = 3z^2\frac{dz}{dx} - z - x\frac{dz}{dx} = 0 = \frac{dz}{dx} = \frac{z}{(3z^2 - x)}$$

$$\frac{\frac{6z^2}{3z^2 - x} - 1}{-(3z^2 - x)^2}$$

$u = \sqrt{xy}$

$v = e^{-xy} + xy$

is there a connection between them? If it's then how?



$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ -ye^{-xy}+y & -xe^{-xy}+X \end{vmatrix} = 0$$

so there is a functional connection between u and v

$u^2 = xy$

$v = e^{-xy} + xy \rightarrow v - e^{-u^2} + u^2 = 0$

$x = r\cos\theta$

$0 = r\cos\theta$

$\theta = 90$ yada $r = 0$

$0 = r\sin\theta$

$\theta = 0$ yada $r = 0$

Directional Derivative

$$\frac{dF}{ds} = \underset{\sim}{\nabla} F \cdot T$$

$F = x^2 y z^3$ $\quad x = e^{-u}$

$y = 2\sin u + 1$

$u = 0$ $\quad z = u - \cos u$

find directional derivative through x, y, z

$$T = \frac{T_0}{|T_0|} \qquad T_0 = \frac{dr}{du}$$

$$\underset{\sim}{\nabla} F = 2xyz^3 \underset{\sim}{i} + x^2 z^3 \underset{\sim}{j} + x^2 y 3z^2 \underset{\sim}{k}$$

$u = 0$

$x = 1$

$y = 1$

$z = -1$

$$\underset{\sim}{\nabla} F \Big|_{u=0} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

$$r = e^{-u}\underset{\sim}{i} + (2\sin u + 1)\underset{\sim}{j} + (u - \cos u)\underset{\sim}{k}$$

$$\frac{dr}{du} = -e^{-u}\underset{\sim}{i} + 2\cos u\,\underset{\sim}{j} + (1 + \sin u)\underset{\sim}{k}$$

$$(-2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) \cdot \left(\frac{-i + 2j + k}{\sqrt{6}}\right)$$

$$T_0\Big|_{u=0} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$= (2 - 2 + 3)\frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

$$T = \frac{-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{6}}$$

$F = 2x^3y - 3y^2z \qquad P(1,2,-1) \qquad Q(3,-1,5)$

Find directional derivative of F for P through Q direction.

$\underset{\sim}{\nabla} F . T \qquad \underset{\sim}{\nabla} F = 6x^2y\,\underset{\sim}{i} + (2x^3 - 6yz)\,j - 3y^2\,\underset{\sim}{k}$

$$\underset{\sim}{\nabla} F\Big|_P = 12\underset{\sim}{i} + 14\underset{\sim}{j} - 12\underset{\sim}{k}$$

$$T_0 = r_q - r_p = (3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$$

$$T = \frac{T_0}{|T_0|} = \frac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{\sqrt{4+9+36}} = \frac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{7}$$

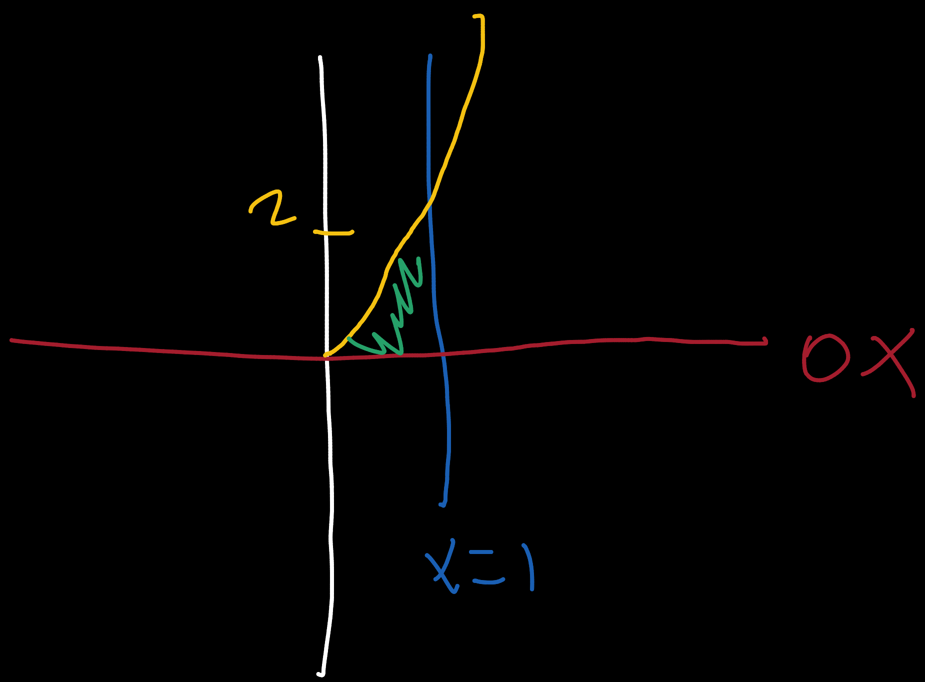
$$\underset{\sim}{\nabla} F . T = (12\underset{\sim}{i} + 14\underset{\sim}{j} - 12\underset{\sim}{k}) . \left(\frac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{7}\right)$$

$$= \frac{24 - 42 - 72}{7} = \frac{-90}{7}$$

what is the max value of directional derivative? $\left|\underset{\sim}{\nabla} F\big|_P\right| = \sqrt{12^2 + 14^2 + (-12)^2}$

Double Integral

$y = 2x^2 \qquad x = 1 \quad \text{and} \quad OX \qquad f(x,y) = x^2y - x + y$



$$\int_0^1 \int_0^{2x^2} x^2 y - x + y \, dy \, dx$$

$$\int_0^1 \frac{x^2 y^2}{2} - xy + \frac{y^2}{2} \Big|_0^{2x^2}$$

$$\frac{x^2(2x^2)^2}{2} - x(2x^2) + \frac{(2x^2)^2}{2}$$

$$\int_0^1 2x^6 - 2x^3 + 2x^4 \, dx$$

$$2\left(\frac{x^7}{7} - \frac{x^4}{4} + \frac{x^5}{5}\right)\Big|_0^1$$

$$2\left(\frac{1}{7} - \frac{1}{4} + \frac{1}{5}\right) = \frac{20 - 35 + 28}{70} = \frac{13}{70}$$

$y \le 2x$, $x^2 - y \le 0$ $\qquad f(x,y) = x^3 + y$

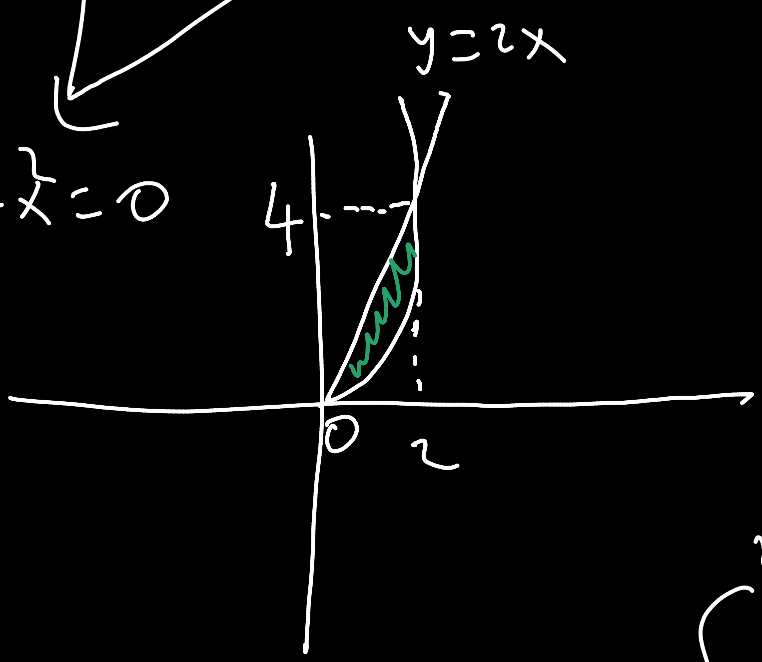
$y = 2x$
but shrinking side

$x^2 = y$
but growing side

$2x - x^2 = 0$

$x = 0$

$x = 2$



$$\int_0^2 \int_{x^2}^{2x} x^3 + y \, dy \, dx$$

$$\int_0^2 x^3 y + \frac{y^2}{2} \Big|_{x^2}^{2x} dx$$

$$\int_0^2 (2x^4 + 2x^2) - \left(x^5 + \frac{x^4}{2}\right) dx$$

$$\frac{2x^5}{5} + \frac{2x^3}{3} - \frac{x^6}{6} - \frac{x^5}{10} \Big|_0^2$$

$$\frac{64}{5} + \frac{16}{3} - \frac{64}{6} - \frac{32}{10}$$

$$= \frac{384 + 160 - 320 - 96}{30} = \frac{128}{30}$$

$$= \frac{64}{15}$$

Area of the defined place

$$\int_0^2 \int_{x^2}^{2x} dy \, dx$$

$$\int_0^2 (2x - x^2) \, dx$$

$$x^2 - \frac{x^3}{3} \Big|_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3} \, br^2$$