$$f(x,y) = 2x^3 + 3xy^2$$

$$\boxed{F_{xy} = \frac{d^2 f}{dy\,dx}}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$\frac{d^2 f}{dx^2} = 12x$$

$$\frac{d^3 f}{dx^3} = 12$$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\frac{df}{dy}$$

$$\frac{df}{dy} = 6xy$$

$$6xy\frac{d}{dx} = 6y$$

$$\frac{df}{dx} = 6xy$$

$$\frac{d^2 f}{dx^2} = 6x$$

$$\frac{d^3 f}{dy^3} = 0$$

$$\frac{d^2 f}{dy\,dx} = \frac{d}{dy}\frac{df}{dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$6x^2 + 3y^2\frac{d}{dy} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$u=(x^2+y^2+z^2)^{-\frac{1}{2}} \Rightarrow \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$$

$$\frac{du}{dx}=2x\cdot\frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dx^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-x\cdot 2x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dy}=2y\cdot\frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-y(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dy^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-y\cdot 2y\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dz}=2z\cdot-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-z\cdot(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dz^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-z\cdot 2z\cdot\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \frac{d^2 z}{d_x dy}\Big|_{(1,1)} = ?$ $\qquad \frac{d}{dx}\left(\frac{dz}{dy}\right) = z_{yx}$

$$\frac{dz}{dy} = x^2\left(\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) = \frac{x \cdot x^2}{x^2+y^2} = \frac{x^3}{x^2+y^2}$$

$$\frac{d}{dx}\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$\frac{d^2 z}{dx\,dy}\Big|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y

$f(x,y)$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$\Delta x \to 0 \quad \Delta y \to 0$

$\Delta x \cong dx \quad \Delta y \cong dy$

$\Delta z = \frac{df}{dx} \cdot \Delta x + \frac{df}{dy} \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$\Delta x \to 0 \quad \Delta y \to 0$

$$\underbrace{\Delta z = \frac{df}{dx} dx + \frac{df}{dy} dy}_{} + \varepsilon_1 dx + \varepsilon_2 dy$$

Important Part = Differentiation of f

$z = f(x,y) = x^2 y - 3y \Rightarrow \Delta z = ? \quad dz = ?$

$\Delta x = -0{,}01 \qquad \Delta y = 0{,}02$

$$\Delta z = \left[(x+\Delta x)^2 (y+\Delta y) - 3(y+\Delta y)\right] - (x^2 y - 3y)$$

$$\left[\left(x^2+2x\Delta x+(\Delta x)^2\right)(y+\Delta y) - 3(y+\Delta y)\right] - y\left(x^2-3\right)$$

$$y+\Delta y \left(x^2+2x\Delta x+(\Delta x)^2-3\right)$$

$$\cancel{yx^2} + y2x\Delta x + y(\Delta x)^2 \cancel{-3y} + \Delta y x^2 + \Delta y 2x\Delta x + \Delta y(\Delta x)^2 - 3\Delta y \cancel{-yx^2} \cancel{+3y}$$

$$\underbrace{2xy\Delta x + x^2\Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x\Delta y + (\Delta x^2)y + \Delta x^2\Delta y}_{\text{ignored}}$$

$$\Delta z = 2xy\Delta x + (x^2-3)\Delta y$$

$$\cong$$

$$dz = 2xy\,dx + x^2 - 3\,dy$$

As we seen here when Δx and Δy go to 0 then $dz \cong \Delta z$

$u = x^2 e^{y/x} \Rightarrow du = ?$ $\quad du = \frac{du}{dx} dx + \frac{du}{dy} dy \ldots\ldots$

$dx = \left(2x e^{y/x} + \frac{y}{x^2} e^{y/x} x^2\right) dx$

$dy = \left(x e^{y/x}\right) dy$

$du = e^{y/x}(2x+y)\,dx + e^{y/x} x \, dy$

Chain Rule

$z = f(x, y) \quad x = g(r, s) \quad y = h(r, s)$

$$\begin{tikzcd}[row sep=small]
 & & r \\
 & x \arrow[ur, no head] \arrow[r, no head] & s \\
z \arrow[ur, no head] \arrow[dr, no head] & & \\
 & y \arrow[r, no head] \arrow[dr, no head] & r \\
 & & s
\end{tikzcd}$$

$$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr}$$

$$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}$$

$z = e^x \sin y$, $x = \ln t$, $y = t^2$ $\quad \frac{dz}{dt} = ?$

$\frac{dz}{dx} = e^x \sin y$

$\frac{dz}{dy} = e^x \cos y$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 2t$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

$$e^x \left(\frac{\sin y}{t} + \cos y \, 2t \right)$$

$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) 2t \right.$$

$$\sin(t^2) + 2t^2 \cos(t^2)$$

$$T = x^3 - xy + y^3 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{dT}{dr} = ?$$

$$\frac{dT}{d\theta} = ?$$

$$\begin{tikzcd}
 & & T \arrow[dl] \arrow[dr] & & \\
 & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\
r & \theta & & r & \theta
\end{tikzcd}$$

$$\frac{dT}{dr} = \frac{dT}{dx} \cdot \frac{dx}{dr} + \frac{dT}{dy} \cdot \frac{dy}{dr}$$

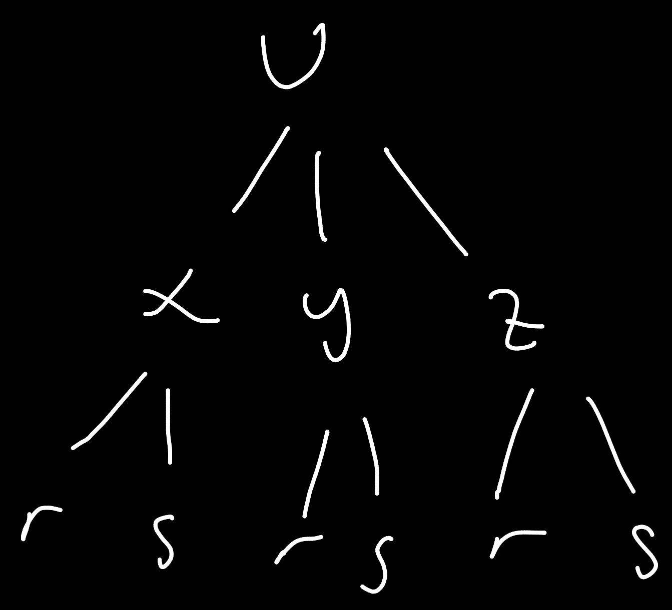
$$\frac{dT}{d\theta} = \frac{dT}{dx} \cdot \frac{dx}{d\theta} + \frac{dT}{dy} \cdot \frac{dy}{d\theta}$$

$$\frac{dT}{dr} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{dT}{d\theta} = (3x^2 - y) \cdot -r\sin\theta + (3y^2 - x)\, r\cos\theta$$

$u = z\sin\left(\frac{y}{x}\right)$, $x = 3r^2 + s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$

$\frac{\partial u}{\partial s} = ?$, $\frac{\partial u}{\partial r} = ?$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$z = f(u) \quad u = x^2 y$ prove that $x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$

$$\begin{array}{ccc} & z & \\ & | & \\ & u & \\ / & & \backslash \\ x & & y \end{array}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{1}{2xy}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial u} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$\frac{1}{2xy}\frac{\partial z}{\partial x} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$$

$x^5 + y = t \qquad u = x^3 y \qquad \frac{du}{dt} = ?$

$x^2 + y^3 = t^2$

$$\begin{tikzcd}[column sep=small] & u \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & y \arrow[d, dash] \\ t & & t \end{tikzcd}$$

$$\frac{du}{dt} = \underbrace{\frac{du}{dx}}_{3x^2y} \cdot \frac{dx}{dt} + \underbrace{\frac{du}{dy}}_{x^3} \cdot \frac{dy}{dt}$$

$$\frac{d}{dt}\left(x^5 + y = t\right) = 5x^4\frac{dx}{dt} + \frac{dy}{dt} = 1$$

$$\frac{d}{dt}\left(x^2 + y^3 = t^2\right) = 2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 2t$$

$$\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$\frac{dx}{dt} = \frac{\begin{bmatrix} 1 & 1 \\ 2t & 3y^2 \end{bmatrix}}{\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix}} = \frac{3y^2 - 2t}{15x^4y^2 - 2x}$$

$$\frac{dy}{dt} = \frac{\begin{vmatrix} 5x^4 & 1 \\ 2x & 2t \end{vmatrix}}{\begin{vmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{vmatrix}} = \frac{10x^4t - 2x}{15x^4y^2 - 2x}$$

$$\frac{du}{dt} = 3x^2y\left(\frac{3y^2 - 2t}{15x^4y^2 - 2x}\right) + x^3\left(\frac{10x^4t - 2x}{15x^4y^2 - 2x}\right)$$

Jacobian

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$F(u,v,w)=0 \quad G(u,v,w)=0 \quad H(u,v,w)=0$$

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$$F(x,y,u,v)=0 \quad G(x,y,u,v)=0$$

$$\frac{du}{dx} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dx} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{du}{dy} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dy} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$u^2 - v = 3x + y$

$u - 2v^2 = x - 2y$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = ?$

$f(u,v,x,y) = 0 \Rightarrow u^2 - v - 3x - y = 0$

$g(u,v,x,y) = 0 \Rightarrow u - 2v^2 - x + 2y = 0$

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{12v - 1}{-8u + 1}$$

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & -1 \\ 2 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4v + 2}{-8u + 1}$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{-2u + 3}{-8u + 1}$$

$$\frac{\partial v}{\partial y} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = -\frac{4u + 1}{-8u + 1}$$

$z^3 - xz - y = 0$ $\qquad \dfrac{d^2 z}{dx\,dy} = ?$

It is closed-form
but we have 2 independent but
1 dependent variable that's why
we can't use Jacobian

$$\frac{dz}{dx\,dy} = \frac{d}{dx}\left(\frac{dz}{dy}\right)$$

$$\frac{dz}{dy} = 3z^2\frac{dz}{dy} - x\frac{dz}{dy} - 1 = 0 \qquad \frac{dz}{dy} = \frac{1}{3z^2 - x}$$

we still don't know

$$\frac{d}{dx}\left(\frac{1}{3z^2 - x}\right) = \frac{d(3z^2-x)^{-1}}{dx} = -(3z^2-x)^{-2}\cdot\left(6z\frac{dz}{dx} - 1\right) = \frac{6z\frac{dz}{dx} - 1}{-(3z^2-x)^2}$$

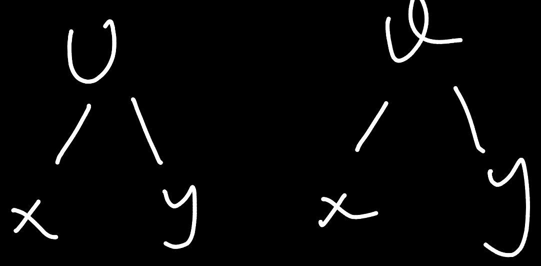
$$\frac{dz}{dx} = 3z^2\frac{dz}{dx} - z - x\frac{dz}{dx} = 0 = \frac{dz}{dx} = \frac{z}{(3z^2-x)}$$

$$\frac{\frac{6z^2}{3z^2-x} - 1}{-(3z^2-x)^2}$$

$u = \sqrt{xy}$ is there a connection

$v = e^{-xy} + xy$ between them?

If it's then how?



$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ -ye^{-xy}+y & -xe^{-xy}+X \end{vmatrix} = 0$$

so there is a functional connection between u and v

$u^2 = xy$

$v = e^{-xy} + xy \rightarrow v - e^{-u^2} + u^2 = 0$

$x = r\cos\theta$

$0 = r\cos\theta$

$\theta = 90$ yada $r = 0$

$0 = r\sin\theta$

$\theta = 0$ yada $r = 0$

Directional Derivative

$$\frac{dF}{ds} = \underset{\sim}{\nabla} F \cdot T$$

$F = x^2 y z^3$

$x = e^{-u}$

$y = 2\sin u + 1$

$z = u - \cos u$

$u = 0$

find directional derivative through x, y, z

$$T = \frac{T_0}{|T_0|} \qquad T_0 = \frac{dr}{du}$$

$$\underset{\sim}{\nabla} F = 2xyz^3 \underset{\sim}{i} + x^2 z^3 \underset{\sim}{j} + x^2 y 3z^2 \underset{\sim}{k}$$

$\underline{u=0}$

$x = 1$

$y = 1$

$z = -1$

$$\underset{\sim}{\nabla} F \Big|_{u=0} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

$$r = e^{-u}\underset{\sim}{i} + (2\sin u + 1)\underset{\sim}{j} + (u - \cos u)\underset{\sim}{k}$$

$$\frac{dr}{du} = -e^{-u}\underset{\sim}{i} + 2\cos u\underset{\sim}{j} + (1 + \sin u)\underset{\sim}{k}$$

$$T_0 \Big|_{u=0} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$T = \frac{-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{6}}$$

$$\left(-2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(\frac{-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{6}}\right)$$

$$= (2 - 2 + 3)\frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

$F = 2x^3y - 3y^2z \qquad P(1,2,-1) \qquad Q(3,-1,5)$

Find directional derivative of F for P through Q direction.

$\underset{\sim}{\nabla} F \cdot T \qquad \underset{\sim}{\nabla} F = 6x^2y\,\underset{\sim}{i} + (2x^3 - 6yz)\,\underset{\sim}{j} - 3y^2\,\underset{\sim}{k}$

$$\underset{\sim}{\nabla} F \Big|_P = 12\underset{\sim}{i} + 14\underset{\sim}{j} - 12\underset{\sim}{k}$$

$$T_0 = r_q - r_p = (3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$$

$$T = \frac{T_0}{|T_0|} = \frac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{\sqrt{4+9+36}} = \frac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{7}$$

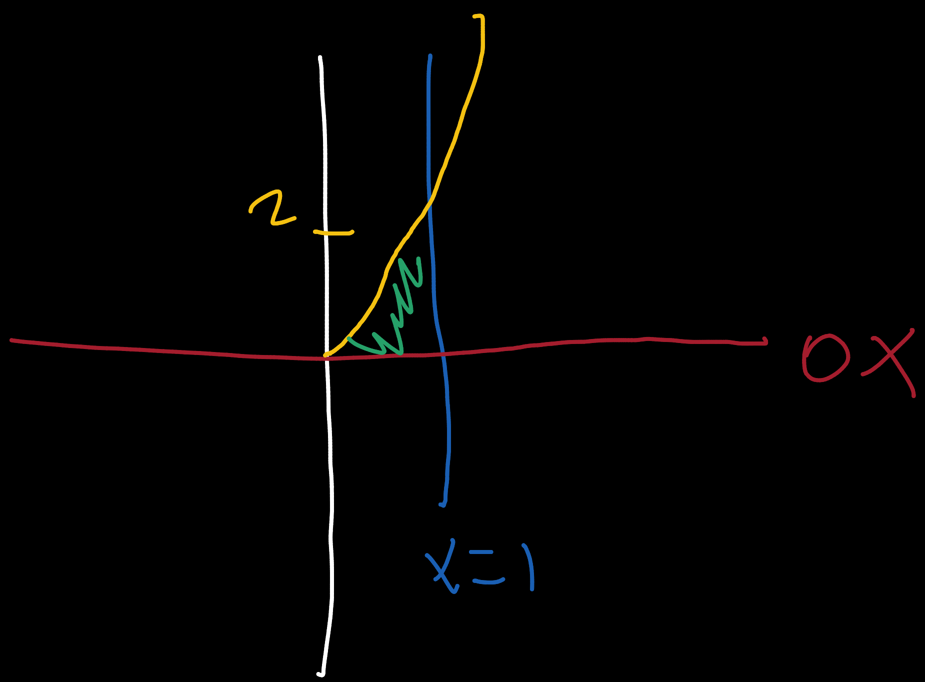
$$\underset{\sim}{\nabla} F \cdot T = (12\underset{\sim}{i} + 14\underset{\sim}{j} - 12\underset{\sim}{k}) \cdot \left(\frac{2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}}{7}\right)$$

$$= \frac{24 - 42 - 72}{7} = \frac{-90}{7}$$

what is the max value of directional derivative? $\left| \underset{\sim}{\nabla} F \big|_P \right| = \sqrt{12^2 + 14^2 + (-12)^2}$

Double Integral

$y = 2x^2 \quad x = 1 \quad \text{and} \quad 0x \qquad f(x,y) = x^2y - x + y$



$$\int_0^1 \int_0^{2x^2} x^2y - x + y \, dy \, dx$$

$$\int_0^1 \frac{x^2y^2}{2} - xy + \frac{y^2}{2} \Big|_0^{2x^2}$$

$$\frac{x^2(2x^2)^2}{2} - x(2x^2) + \frac{(2x^2)^2}{2}$$

$$\int_0^1 2x^6 - 2x^3 + 2x^4 \, dx$$

$$2\left(\frac{x^7}{7} - \frac{x^4}{4} + \frac{x^5}{5}\right)\Big|_0^1$$

$$2\left(\frac{1}{7} - \frac{1}{4} + \frac{1}{5}\right) = \frac{20 - 35 + 28}{70} = \frac{13}{70}$$

$y \le 2x \quad , \quad x^2 - y \le 0 \qquad f(x,y) = x^3 + y$

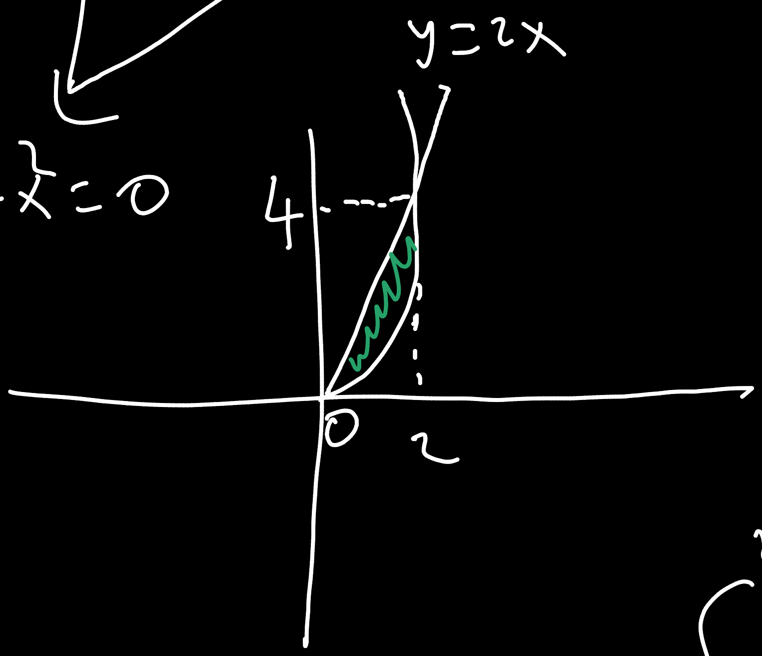
$y = 2x$
but shrinking side

$x^2 = y$
but growing side

$2x - x^2 = 0$

$x = 0$

$x = 2$



Area of the defined place

$$\int_0^2 \int_{x^2}^{2x} dy\,dx$$

$$\int_0^2 (2x - x^2)\,dx$$

$$x^2 - \frac{x^3}{3} \Big|_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$ br

$$\int_0^2 \int_{x^2}^{2x} x^3 + y \; dy\,dx$$

$$\int_0^2 x^3 y + \frac{y^2}{2} \Big|_{x^2}^{2x} dx$$

$$\int_0^2 (2x^4 + 2x^2) - \left(x^5 + \frac{x^4}{2}\right) dx$$

$$\frac{2x^5}{5} + \frac{2x^3}{3} - \frac{x^6}{6} - \frac{x^5}{10} \Big|_0^2$$

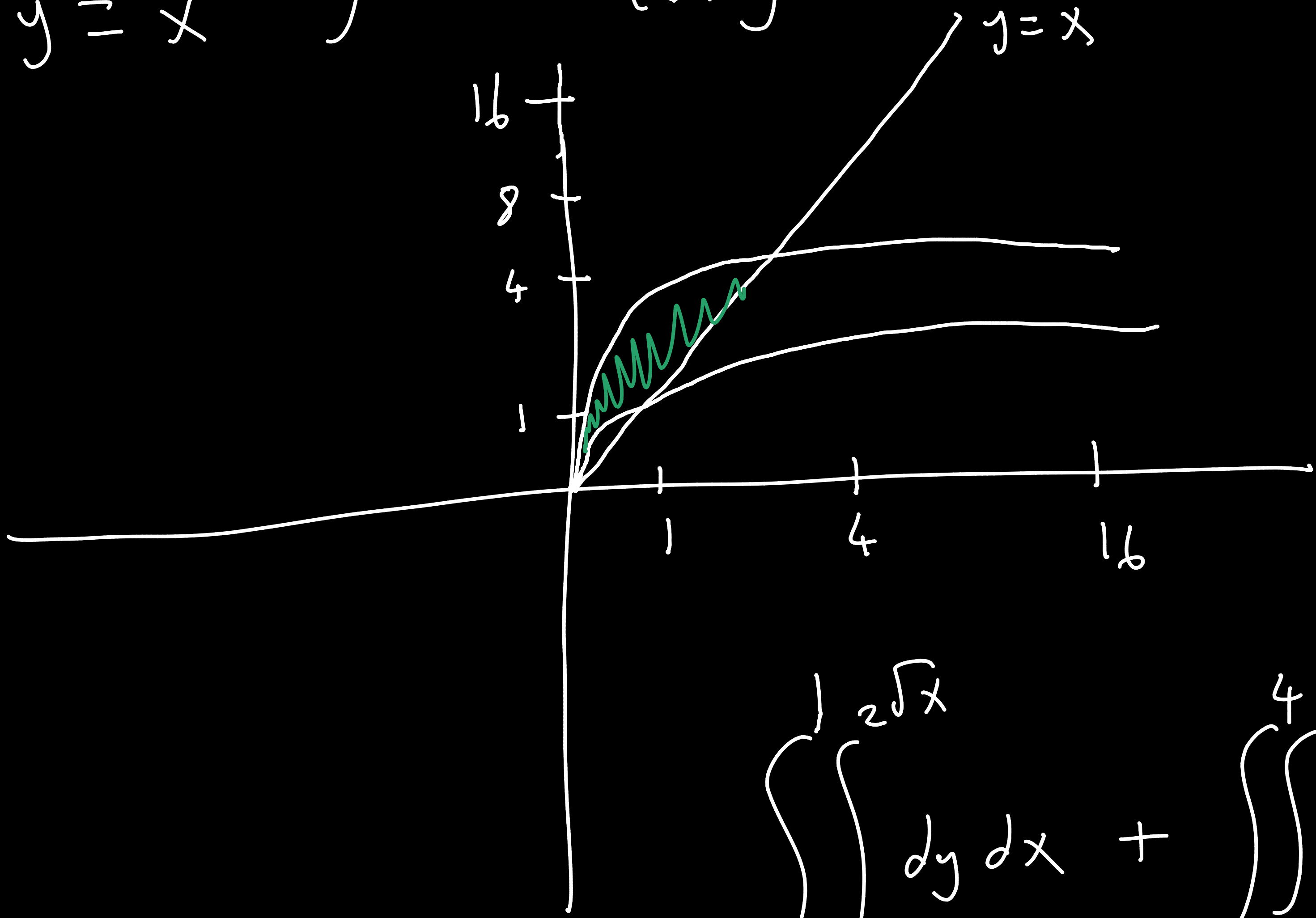
$$\frac{64}{5} + \frac{16}{3} - \frac{64}{6} - \frac{32}{10}$$

$$= \frac{384 + 160 - 320 - 96}{30} = \frac{128}{30}$$

$$= \frac{64}{15}$$

$y = 2\sqrt{x}$
$y = \sqrt{x}$
$y = x$

Area of ~~the~~ this conjunction



$$\int_0^1 \int_{\sqrt{x}}^{2\sqrt{x}} dy\,dx + \int_1^4 \int_x^{2\sqrt{x}} dy\,dx$$

$$\int_0^1 \left(2\sqrt{x} - \sqrt{x}\right) dx + \int_1^4 \left(2\sqrt{x} - x\right) dx$$

$$\left.\frac{2x^{\frac{3}{2}}}{3}\right|_0^1 + \left.\frac{4x^{\frac{3}{2}}}{3} - \frac{x^2}{2}\right|_1^4$$

$$= \frac{2}{3} + \left(\frac{32}{3} - 8\right) - \left(\frac{4}{3} - \frac{1}{2}\right)$$

$$= \frac{2}{3} + \frac{8}{3} - \frac{5}{6} = \frac{4}{6} + \frac{16}{6} - \frac{5}{6} = \frac{15}{6} = \frac{5}{2}$$

$x^2+y^2 \le 4x$ area of this?

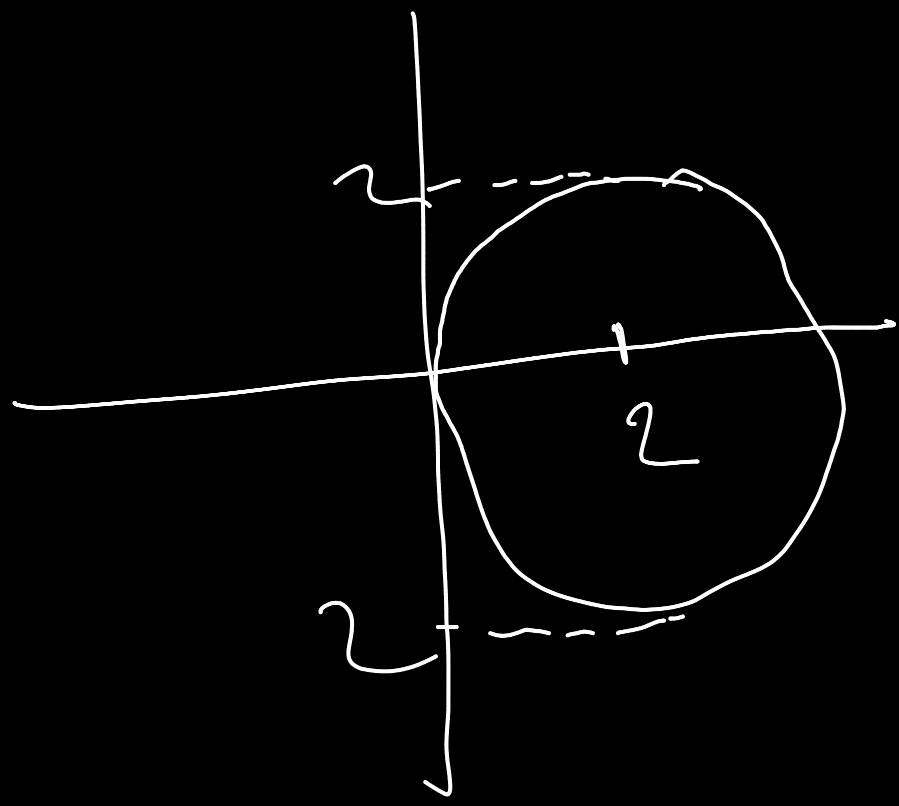
$x^2+y^2-4x=0$

$x^2+y^2-4x+4-4=0$

$x^2-4x+4+y^2=4$

$(x-2)^2+y^2=4$

$(x-a)^2+(y-b)^2=r^2$



$y=\sqrt{4x-4}$

$y=\sqrt{4-(x-2)^2}$ we choose this for easier integ

$x-2=u$ $dx=du$

$$\frac{A}{2}=\int_0^4\int_0^{\sqrt{4-(x-2)^2}} dy\,dx=\int_{-2}^{2}\sqrt{4-u^2}\,du=\int_{-2}^{2}\sqrt{2^2-u^2}\,du$$

$\sqrt{a^2-x^2} \to x=a\sin t$

$u = 2\sin t$

$du = 2\cos t\,dt$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4-(2\sin t)^2}\; 2\cos t\,dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2 t)}\; 2\cos t\,dt$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos 2t\,dt$$

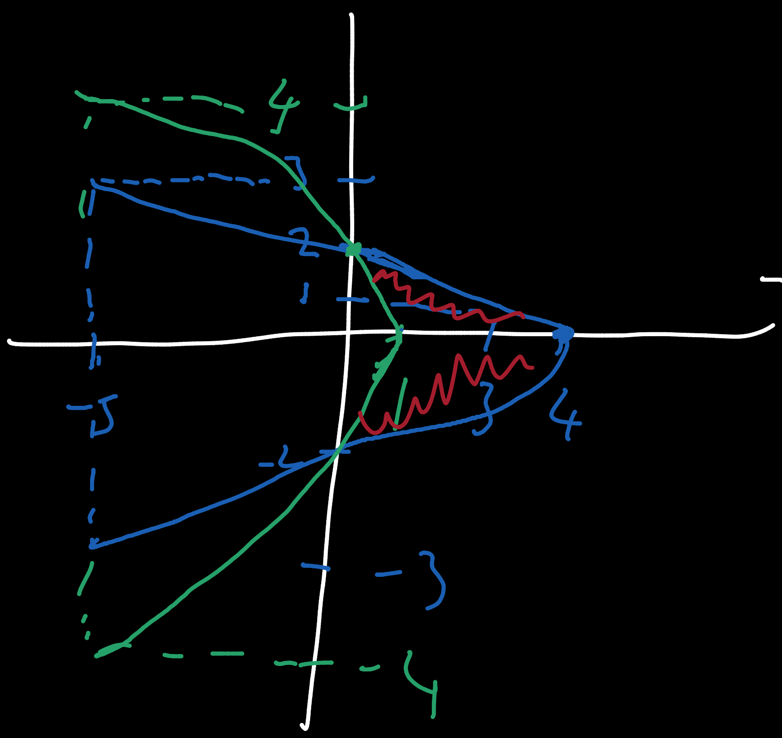
$$2\left(t+\frac{\sin 2t}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = 2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 2\pi = \frac{A}{2}$$

$$A = 4\pi$$

$y^2 = 4 - x$

$y^2 = 4 - 4x$ Area ?

$x = \frac{4-y^2}{4}$ $\quad x = 4 - y^2$



$$\int_{-2}^{2}\int_{\frac{4-y^2}{4}}^{4-y^2} dx\,dy$$

$$= \int_{-2}^{2} 4 - y^2 - \left(\frac{4-y^2}{4}\right) dy$$

$$= \frac{1}{4}\int_{-2}^{2} 16 - 4y^2 - 4 + y^2 \, dy$$

$$= \frac{1}{4}\int_{-2}^{2} 12 - 3y^2 \, dy$$

$$= \frac{1}{4}\left(12y - y^3\right)\Big|_{-2}^{2}$$

$$= \frac{1}{4}(24 - 8) - (-24 + 8)$$

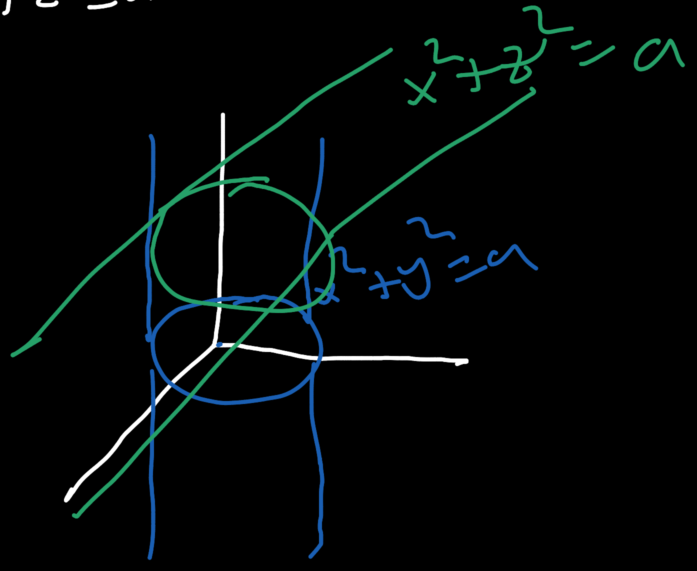
$$= 8 \text{ Area}$$

$x^2+y^2=a^2$

$x^2+z^2=a^2$

Volume?

$z=\sqrt{a^2-x^2}$ $\quad$ $y=\sqrt{a^2-x^2}$



$$V=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy\,dx$$

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,y\Big|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}=2\int_{-a}^{a}a^2-x^2\,dx$$

$$=2\left(\left(a^2x-\frac{x^3}{3}\right)\right)\Big|_{-a}^{a}=2\left(\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)\right)$$

$$=2\left(2a^3-\frac{2a^3}{2}\right)=4a^3-\frac{4a^3}{3}=\frac{8a^3}{3}$$

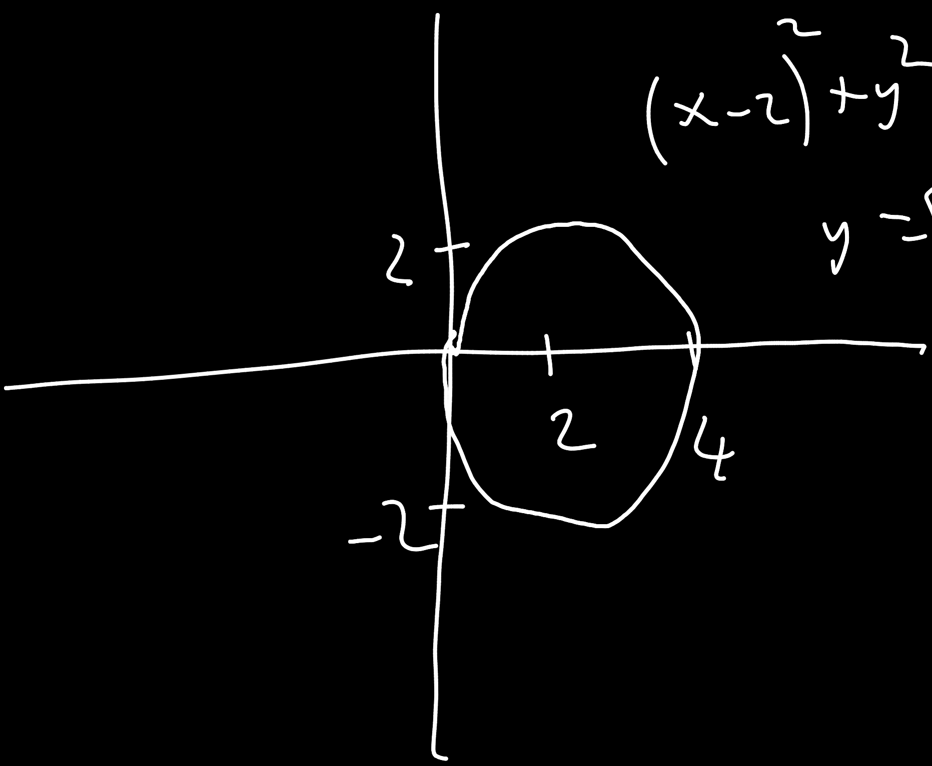
$z=0 \qquad x^2+y^2 \le 4x \qquad z+x=8 \qquad$ Volume?

$$y^2 = 4x - x^2$$

$$y = \pm\sqrt{4x-x^2}$$

$(x-2)^2+y^2=4$

$y=\sqrt{4-(x-2)^2}$



$$\int_0^4 \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} (8-x)\,dy\,dx$$

$$\int_0^4 (8y-xy)\Big|_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} dx = \int_0^4 8\sqrt{4-(x-2)^2} - x\sqrt{4-(x-2)^2} - \left(-8\sqrt{4-(x-2)^2} + x\sqrt{4-(x-2)^2}\right) dx$$

$$= \int_0^4 16\sqrt{4-(x-2)^2} - 2x\sqrt{4-(x-2)^2} = 2\int_0^4 (8-x)\left(\sqrt{4-(x-2)^2}\right) dx$$

$\sqrt{a^2-x^2}$

$x = a\sin t$

$dx = a\cos t\,dt$

$$16\int_0^4 \sqrt{4-(x-2)^2}\,dx \quad - \quad 2\int_0^4 x\sqrt{4-(x-2)^2}\,dx$$

$\perp \quad x-2 = 2\sin t \qquad \perp \quad x-2=u, \ dx=du$

$$16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2 t)}\,2\cos t\,dt \qquad -2\int_{-2}^{2}(u+2)\sqrt{2^2-u^2}\,du$$

$$64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\,dt \qquad -2\int_{-2}^{2} u\sqrt{4-u^2}\,du - 4\int_{-2}^{2}\sqrt{4-u^2}\,du$$

$u = 2\sin m$

$du = 2\cos m\,dm$

$$-4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2 m}\,dm$$

$4-u^2 = p$

$-2u\,du = dp$

$$\int_0^0 p^{\frac{1}{2}}dp = \frac{2p^{\frac{3}{2}}}{3} = \frac{2(4-u^2)^{\frac{3}{2}}}{3}\Big|_{u=-2}^{u=2} = 0$$

$$64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt$$

$$= -16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 m\,dm = -8\left(m+\cos 2m\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = -8\pi$$

$$32\left(t+\frac{\sin 2t}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$= 32\left[\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right]$$

$$= 32\pi$$

24π

Transformation with Double Integrals

$x^2-y^2=1$

$x^2-y^2=9$

$xy=2$

$xy=4$

$\iint (x^2+y^2)\,dx\,dy = ?$

$x^2-y^2=u$

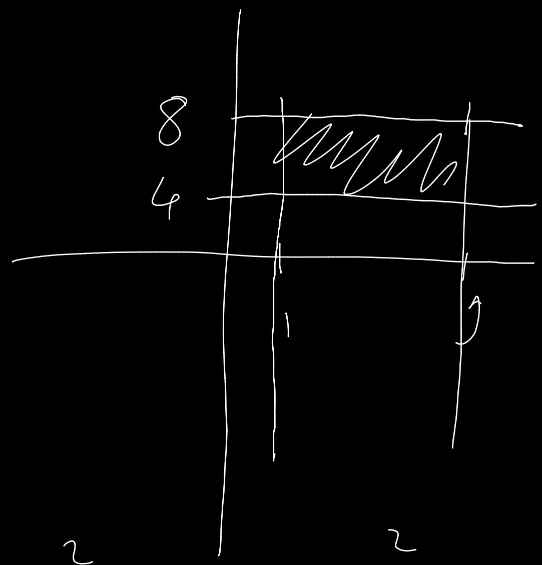
$u=1$

$u=9$

$2xy=v$

$v=4$

$v=8$



$$u^2+v^2=(x^2)^2-2x^2y^2+(y^2)^2+4x^2y^2$$
$$=(x^2+y^2)^2$$
$$x^2+y^2=\sqrt{u^2+v^2}$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4x^2+4y^2 \qquad J=\frac{1}{4x^2+4y^2}=\frac{1}{4\sqrt{u^2+v^2}}$$

$$\int_1^9\int_4^8 \sqrt{u^2+v^2}\,\frac{1}{4\sqrt{u^2+v^2}}\,dv\,du=\frac{1}{4}\int_1^9 4\,du = 8$$

$x^2+y^2=4$

$x^2+y^2=9$

$\iint \sqrt{x^2+y^2}\,dx\,dy = ?$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$r = \sqrt{x^2+y^2}$

$r=2 \quad \theta=0$

$r=3 \quad \theta=2\pi$

$$J = \frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta+\sin^2\theta) = r$$

$$\int_0^{2\pi}\int_2^3 r \cdot r \, dr\, d\theta$$

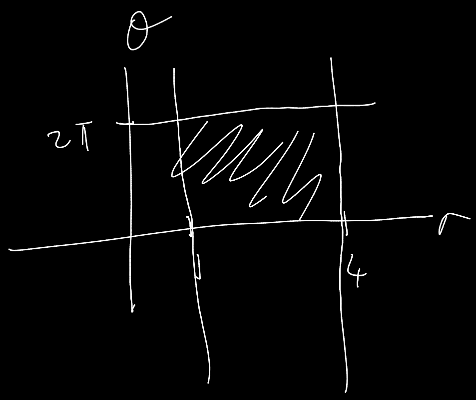
$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_2^3 d\theta = \int_0^{2\pi}\left(9-\frac{8}{3}\right)d\theta = \int_0^{2\pi}\frac{19}{3}\,d\theta$$

$$= \frac{38\pi}{3}$$

$x^2+y^2=1 \quad r=1$ $\quad x=r\cos\theta \quad y=r\sin\theta \quad \frac{x}{y}=\tan\theta \quad J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}=r$

$x^2+y^2=16 \quad r=4$

$\iint (x^2+y^2)^{\frac{1}{4}}\,dx\,dy \Big\}_1$



$x^2+y^2=r^2\cos^2\theta+r^2\sin\theta=r^2$

$(x^2+y^2)^{\frac{1}{4}}=\sqrt{r}$

$$\int_0^{2\pi}\int_1^4 \sqrt{r}\,r\,dr\,d\theta=\int_0^{2\pi}\int_1^4 r^{\frac{3}{2}}\,dr\,d\theta$$

$$\int_0^{2\pi} \frac{2r^{\frac{5}{2}}}{5}\Big|_1^4 d\theta=\frac{2}{5}\int_0^{2\pi} 31\,d\theta$$

$$=\frac{124\pi}{5}$$

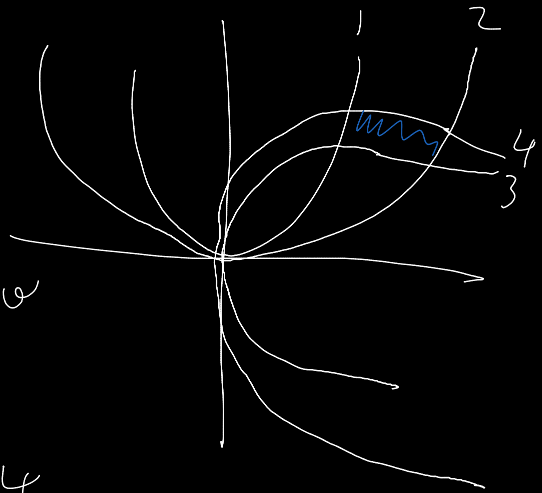
$y = x^2$

$4y = x^2$

$2x = y^2$

$3x = y^2$

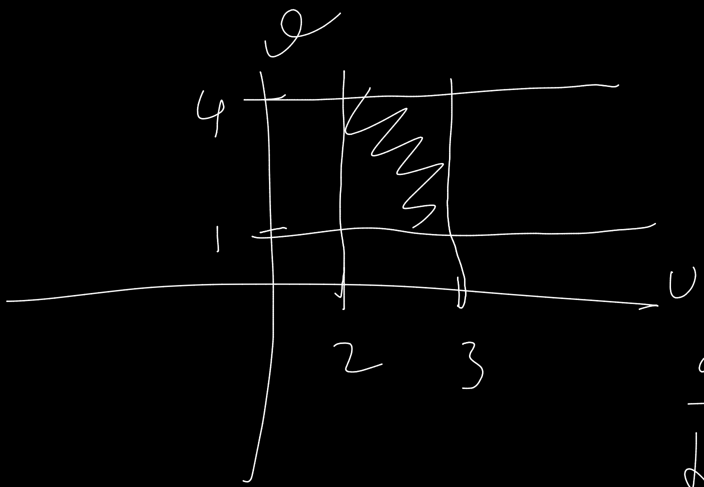
$$\iint_B dx\,dy = ?$$



$\frac{y^2}{x} = u$ $\quad \frac{x^2}{y} = v$

$u = 2$ $\quad v = 1$

$u = 3$ $\quad v = 4$



$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y^2x^{-2} & 2yx^{-1} \\ 2xy^{-1} & -x^2y^{-2} \end{vmatrix} =$$

$$= \left(x^2 \cdot \frac{1}{x^2} \cdot y^2 \cdot \frac{1}{y^2}\right) - \left(4y \cdot \frac{1}{y} \cdot x \cdot \frac{1}{x}\right)$$

$$= 1 - 4 = 3$$

$$|J| = \frac{1}{3}$$

$$\int_2^3 \int_1^4 \frac{1}{3}\, dv\, du$$

$$= \int_2^3 \frac{v}{3}\Big|_1^4 du$$

$$= \int_2^3 1\, du$$

$$= 1$$

$y = 2x - 2$ $\qquad u = x + y$ $\qquad u = 2 \quad w = 2$

$y = 2x - 4$ $\qquad w = 2x - y$ $\qquad u = 4 \quad w = 4$

$y = -x + 2$

$y = -x + 4$

$$\iint\limits_{B} \underbrace{(2x - y)^2}_{w^2} \, dx\, dy \, ?, \qquad \frac{\partial(x,y)}{\partial(u,w)} \cdot \frac{\partial(u,w)}{\partial(x,y)} = 1$$

$$\frac{\partial(u,w)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ w_x & w_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$\int_2^4 \int_2^4 \frac{w^2}{3} \, dw \, du = \frac{1}{3} \int_2^4 \frac{w^3}{3} \Big|_2^4 du = \frac{1}{9} \int_2^4 (64 - 8) \, du$$

$$\frac{1}{9} \left(56u \Big|_2^4 \right) = \frac{1}{9} (224 - 112) = \frac{112}{9}$$

$$\iint\limits_{B} dx\, dy = \iint\limits_{B'} |J| \, du \, dw$$

Mess Center of the Mess

$M = \iint d(x,y)\,dx\,dy$

$\bar{x} = \frac{1}{M}\iint d(x,y)\,x\,dx\,dy$

Moment of Inertia

$\bar{y} = \frac{1}{M}\iint d(x,y)\,y\,dx\,dy$

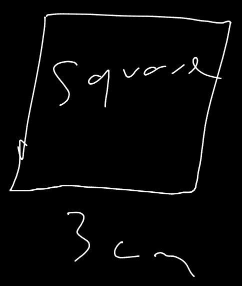
$I = \iint d(x,y)\,l^2\,dx\,dy$

Density is changing correspondes to square of distance between a cornor

Mess?

Center of the mess?

Moment of inertia correspondes to $x=4$ line?



$$M = \int_0^3\int_0^3 k\, x^2+y^2\, dy\,dx = k\int_0^3\int_0^3 x^2+y^2\,dy\,dx = k\int_0^3 x^2y+\frac{y^3}{3}\Big|_0^3 dx$$

$$= k\int_0^3 3x^2+9\,dx = 3k\left(\frac{x^3}{3}+3x\Big|_0^3\right) = 54k$$

$$\bar{x} = \frac{1}{54k}\int_0^3\int_0^3 k(x^2+y^2)\,x\,dx\,dy = \frac{1}{54}\int_0^3 \frac{x^4}{4}+\frac{y^2x^2}{2}\Big|_0^3 dy = \frac{1}{54}\int_0^3 \frac{81}{4}+\frac{9y^2}{2}\,dy = \frac{1}{54}\left(\frac{81.3}{4}+\frac{9.9}{2}\right) = \frac{3^5+2.3^4}{4.27.2} = \frac{3^4(5)}{8.3^3} = \frac{15}{8}$$

$$\bar{y} = \frac{1}{54k}\int_0^3\int_0^3 k(x^2+y^2)\,y\,dx\,dy = \frac{1}{54}\int_0^3\int_0^3 (x^2y+y^3)\,dx\,dy = \frac{1}{54}\int_0^3 \frac{x^3y}{3}+xy^3\Big|_0^3 dy = \frac{1}{54}\int_0^3\left(\frac{27y}{3}+3y^3\right)dy = \frac{1}{18}\int_0^3(3y+y^3)\,dy = \frac{1}{18}\left(\frac{3y^2}{2}+\frac{y^4}{4}\Big|_0^3\right) = \frac{1}{18}\left(\frac{27}{2}+\frac{81}{4}\right) = \frac{2.3^3+3^4}{3^2.2.4} = \frac{3^3(5)}{3^2.2.4} = \frac{15}{8}$$

$$I = \int_0^3\int_0^3 k(x^2+y^2)(4-x)^2\,dy\,dx = k\int_0^3\int_0^3 (x^2+y^2)(16-8x+x^2)\,dy\,dx = k\int_0^3\int_0^3\left(16x^2+16y^2-8x^3-8xy^2+x^4+x^2y^2\right)dy\,dx =$$

$16x^2y+16\frac{y^3}{3}$

$-8x^3y-8x\frac{y^3}{3}$

$x^4y+x^2\frac{y^3}{3}$

$$48k\int_0^3\left(x^2+3\right)dx = 48k\left(\frac{x^3}{3}+3x\Big|_0^3\right) = 48k.18$$

$$-24k\int_0^3\left(x^3+3x\right)dx = -24k\left(\frac{x^4}{4}+\frac{3x^2}{2}\Big|_0^3\right) = -6k.\left(3^4+2.3^3\right)$$

$$3k\int_0^3\left(x^4+3x^2\right)dx = 3k\left(\frac{x^5}{5}+x^3\Big|_0^3\right) = \frac{3}{5}k\left(3^5+5.3^3\right)$$

$$\frac{1404k}{5}$$

$y^2 = 4x+4$

$y^2 = 4-2x$

homogen plate.

center of the mass

moment of inertia

based on:

X axis

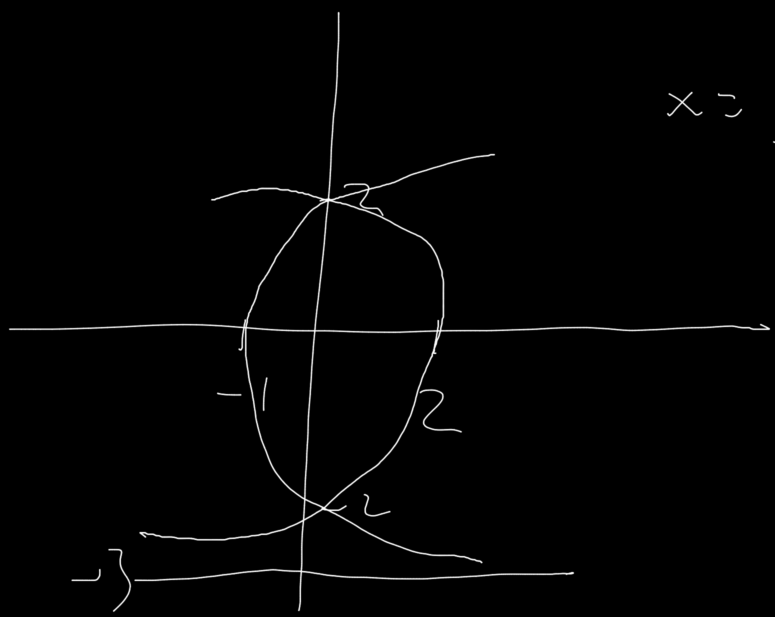
$y=-3$ line

$$4x+4 = 4-2x$$

$$6x = 0 \qquad y^2 = 4 \quad y=2$$

$$y=-2$$

$$x = \frac{y^2-4}{4} \qquad x = \frac{4-y^2}{2}$$



$$\bar{x} = \frac{1}{m}\int x \cdot \delta(x,y)\, dx\, dy$$

$$\bar{y} = \frac{1}{m}\int y \cdot \delta(x,y)\, dx\, dy$$

$$A = \int_{-2}^{2}\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} dx\,dy = \int_{-2}^{2} \frac{8-2y^2-y^2+4}{4}\, dy$$

$$= \frac{3}{4}\int_{-2}^{2} 4-y^2\, dy = \frac{3}{4}\left(4y - \frac{y^3}{3}\Big|_{-2}^{2}\right)$$

$$= \frac{3}{4}\left(8-\frac{8}{3}+8-\frac{8}{3}\right) = \left(\frac{48-16}{3}\right)\cdot\frac{3}{4}$$

$$A = \frac{32\cdot 3}{3\cdot 4} = 8 \qquad m = 8k$$

$$\bar{x} = \frac{1}{8k}\int_{-2}^{2}\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} x \cdot k \cdot dx\,dy$$

$$= \frac{1}{16}\int_{-2}^{2}\left(\frac{4-y^2}{2}\right)^2 - \left(\frac{y^2-4}{4}\right)^2 dy$$

$$= \frac{1}{16}\int_{-2}^{2}\left(\frac{16-8y^2+y^4}{4} - \frac{y^4-8y^2+16}{16}\right) dy$$

$$= \frac{1}{16.16}\int_{-2}^{2}\left(64-32y^2+4y^4-y^4+8y^2-16\right) dy$$

$$= \frac{1}{16.16}\int_{-2}^{2}\left(48-24y^2+3y^4\right) dy$$

$$= \frac{1}{16.16}\left(48y - 8y^3 + \frac{3y^5}{5}\Big|_{-2}^{2}\right)$$

$$= \frac{1}{16.16}\left(96-64+\frac{3.32}{5} - \left(-96+64-\frac{3.32}{5}\right)\right)$$

$$\frac{1}{16.16}\left(96+96-64-64+\frac{3.32}{5}+\frac{3.32}{5}\right) = \frac{1}{16.16}\left(64+2\left(\frac{3.32}{5}\right)\right) = \frac{1}{16.16}\, 64\left(\frac{4.2}{5}\right)$$

$$\bar{x} = \frac{2}{5}$$

$$\bar{y} = \frac{1}{8k}\int_{-2}^{2}\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} y \cdot k \cdot dx\,dy = \frac{1}{8}\int_{-2}^{2} y\left(\frac{4-y^2}{2} - \frac{y^2-4}{4}\right) dy$$

$$= \frac{1}{8.4}\int_{-2}^{2}(8-2y^2-y^2+4)\,y\,dy = \frac{3}{8.4}\int_{-2}^{2} 4y-y^3\, dy = \frac{3}{8.4}\left(2y^2 - \frac{y^4}{4}\Big|_{-2}^{2}\right)$$

$$= \frac{3}{8.4}\left[(8-4)-(8-4)\right] = 0 = \bar{y}$$

$$I = \iint \delta(x,y)\, l^2\, dx\,dy$$

$$\int_{-2}^{2}\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} k \cdot y^2\, dx\,dy = k\int_{-2}^{2} y^2\left(\frac{4-y^2}{2} - \frac{y^2-4}{4}\right) dy$$

$$= \frac{k}{4}\int_{-2}^{2}(8-2y^2-y^2+4)\,y^2\,dy = \frac{3k}{4}\int_{-2}^{2} 4y^2-y^4\, dy = \frac{3k}{4}\left(\frac{4}{3}y^3 - \frac{y^5}{5}\Big|_{-2}^{2}\right)$$

$$= \frac{3k}{4}\left[\left(\frac{32}{3}-\frac{32}{5}\right) - \left(\frac{-32}{3}+\frac{32}{5}\right)\right] = \frac{3k}{4}\left(\frac{64}{3}-\frac{64}{5}\right)$$

$$= \frac{3k}{4.15}\left(4\left(\frac{16}{3}-\frac{16}{5}\right)\right) = \frac{k}{5}\cdot\left(16.5-16.3\right) = \frac{k}{5}\left(16(5-3)\right)$$

$$= \frac{32k}{5}\ kr^4$$

$$\int_{-2}^{2}\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} k(y+3)^2\, dx\,dy = k\int_{-2}^{2}\int_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} y^2+6y+9\, dx\,dy = k\int_{-2}^{2}(y^2+6y+9)\, x\Big|_{\frac{y^2-4}{4}}^{\frac{4-y^2}{2}} dy$$

$$k\int_{-2}^{2}(y^2+6y+9)\left(\frac{4-y^2}{2} - \frac{y^2-4}{4}\right) dy = \frac{3k}{4}\int_{-2}^{2}(y^2+6y+9)(4-y^2)\, dy$$

$$= \frac{3k}{4}\int_{-2}^{2}\left(4y^2-y^4+24y-6y^3+36-9y^2\right) dy = \frac{3k}{4}\int_{-2}^{2}\left(-y^4-6y^3-5y^2+24y+36\right) dy$$

$$= \frac{3k}{4}\left(\frac{-y^5}{5} - \frac{3y^4}{2} - \frac{5y^3}{3} + 12y^2 + 36y\Big|_{-2}^{2}\right) = \frac{3k}{4}\left(\left(\frac{-32}{5}-24-\frac{40}{3}+48+72\right) - \left(\frac{32}{5}-24+\frac{40}{3}+48-72\right)\right)$$

$$= \frac{3k}{4}\left(\frac{-64}{5}+0-\frac{80}{3}+0+144\right)$$

$$= k\left(\frac{3.-4.4}{4.5} - \frac{3.\ 4.20}{4.3} + \frac{3.36.4}{4}\right) = k\left(\frac{-48}{5} - 20 + 108\right) = k\left(\frac{-48}{5}+88\right)$$

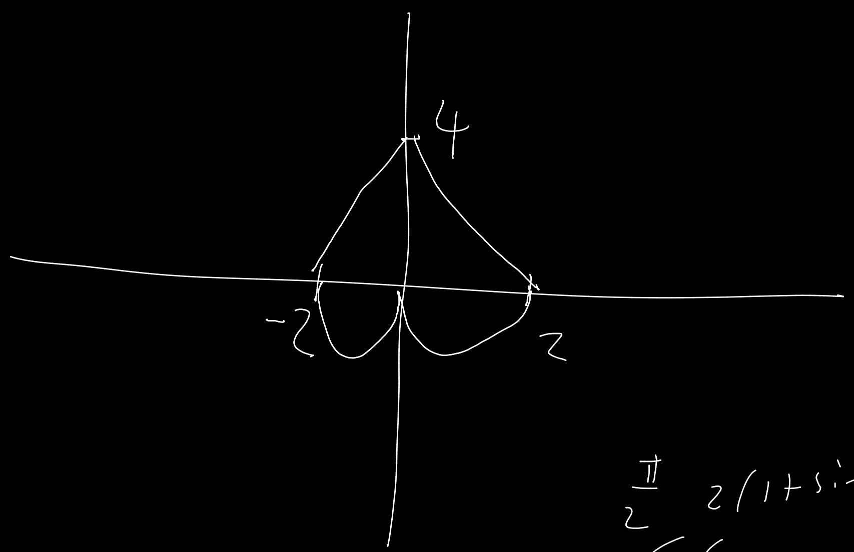
$$= \frac{(-48+440)k}{5} = \frac{392k}{5}$$

$r = 2(1+\sin\theta)$ density = each point oppositely relayted to origin

Use only right half

$m = ?$

Center of $m = ?$



$$m = \iint_{B} d(x,y)\,dx\,dy$$

$$d(x,y) = \frac{k}{\sqrt{x^2+y^2}}$$

$$d(r,\theta) = \frac{k}{r}$$

$$dx\,dy = |J|\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$$J = \frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2(1+\sin\theta)} \frac{k}{r}\, r \, dr \, d\theta$$

$$k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2+2\sin\theta \, d\theta$$

$$= 2k \left(\theta - \cos\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$= 2k\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{-\pi}{2} - 0\right)\right] = 2\pi k$$

$$\bar{x} = \frac{1}{2\pi k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2(1+\sin\theta)} \frac{k}{r}\, r\cos\theta \, r\,dr\,d\theta$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2+2\sin\theta} r\cos\theta \, dr\,d\theta$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2+2\sin\theta)^2}{2} \cos\theta\, d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \cos\theta \, d\theta$$

$1+\sin\theta = u$

$\cos\theta\, d\theta = du$

$$= \frac{1}{\pi} \int_0^2 u^2\, du = \frac{1}{\pi}\left(\frac{u^3}{3} \Big|_0^2\right) = \frac{8}{3\pi} = \bar{x}$$

$$\bar{y} = \frac{1}{2k\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2(1+\sin\theta)} \frac{k}{r} \cdot r\sin\theta \cdot r\,dr\,d\theta = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(r^2\sin\theta \Big|_0^{2(1+\sin\theta)} d\theta \right)$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4(1+\sin\theta)^2}{2} \sin\theta \, d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \sin\theta\, d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+2\sin\theta + \sin^2\theta)\sin\theta\, d\theta, \ \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, d\theta = 0$$

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3\theta\, d\theta$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta)\, d\theta$$

$$\hookrightarrow \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, d\theta = 0$$

$\cos\theta = u$

$-\sin\theta\, d\theta = du$

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin\theta\cos^2\theta\, d\theta$$

$$= \frac{1}{\pi} \int_0^0 u\, du = 0$$

$$\frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\, d\theta$$

$$= \frac{1}{\pi}\left(\theta - 2\sin 2x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$$\text{Center} = \left(\frac{8}{3\pi}, 1\right)$$