$$f(x,y) = 2x^3 + 3xy^2$$

$$\boxed{F_{xy} = \frac{d^2 f}{dy\,dx}}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$\frac{d^2 f}{dx^2} = 12x$$

$$\frac{d^3 f}{dx^3} = 12$$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\frac{df}{dy}$$

$$\frac{df}{dy} = 6xy$$

$$6xy\frac{d}{dx} = 6y$$

$$\frac{df}{dx} = 6xy$$

$$\frac{d^2 f}{dx^2} = 6x$$

$$\frac{d^3 f}{dy^3} = 0$$

$$\frac{d^2 f}{dy\,dx} = \frac{d}{dy}\frac{df}{dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$6x^2 + 3y^2\frac{d}{dy} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$u=(x^2+y^2+z^2)^{-\frac{1}{2}} \Rightarrow \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$$

$$\frac{du}{dx}=2x\cdot\frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dx^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-x\cdot2x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dy}=2y\cdot\frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-y(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dy^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-y\cdot2y\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dz}=2z\cdot-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}=-z\cdot(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dz^2}=-(x^2+y^2+z^2)^{-\frac{3}{2}}-z\cdot2z\cdot\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \frac{\partial^2 z}{\partial x \partial y}\Bigg|_{(1,1)} = ? \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{yx}$$

$$\frac{\partial z}{\partial y} = x^2 \left(\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} \right) = \frac{x . x^2}{x^2+y^2} = \frac{x^3}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y}\Bigg|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y

$f(x,y)$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$\Delta x \to 0 \quad \Delta y \to 0$

$\Delta x \cong dx \quad \Delta y \cong dy$

$$\Delta z = \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$\Delta x \to 0 \quad \Delta y \to 0$

$$\Delta z = \underbrace{\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}_{\text{Important Part = Diffentiation of } f} + \varepsilon_1 dx + \varepsilon_2 dy$$

$z = f(x,y) = x^2 y - 3y \Rightarrow \Delta z = ? \quad dz = ?$

$\Delta x = -0{,}01 \qquad \Delta y = 0{,}02$

$$\Delta z = \left[(x+\Delta x)^2 (y+\Delta y) - 3(y+\Delta y)\right] - (x^2 y - 3y)$$

$$\left[(x^2 + 2x\Delta x + (\Delta x)^2)(y+\Delta y) - 3(y+\Delta y)\right] - y(x^2 - 3)$$

$$y + \Delta y \left(x^2 + 2x\Delta x + (\Delta x)^2 - 3\right)$$

$$\cancel{yx^2} + y2x\Delta x + y(\Delta x)^2 \cancel{-3y} + \Delta y x^2 + \Delta y 2x \Delta x + \Delta y (\Delta x)^2 - 3\Delta y \cancel{-yx^2} \cancel{+3y}$$

$$\underbrace{2xy\Delta x + x^2 \Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x \Delta y + (\Delta x^2) y + \Delta x^2 \Delta y}_{\text{ignored}}$$

$$\Delta z = 2xy\Delta x + (x^2 - 3)\Delta y$$

$$\cong$$

$$dz = 2xy\,dx + x^2 - 3\,dy$$

As we seen here when Δx and Δy go to 0 then $dz \cong \Delta z$

$u = x^2 e^{y/x} \Rightarrow du = ?$ $\qquad du = \frac{du}{dx} dx + \frac{du}{dy} dy \ldots\ldots$

$dx = \left(2x e^{y/x} + \frac{y}{x^2} e^{y/x} x^2\right) dx$

$dy = \left(x e^{y/x}\right) dy$

$$du = e^{y/x}(2x+y)\,dx + e^{y/x} x \, dy$$

Chain Rule

$z = f(x,y) \qquad x = g(r,s) \qquad y = h(r,s)$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & x \arrow[r] \arrow[dr] & r \\
z \arrow[ur] \arrow[dr] & & s \\
 & y \arrow[r] \arrow[dr] & r \\
 & & s
\end{tikzcd}$$

$$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr}$$

$$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}$$

$z = e^x \sin y \quad , x = \ln t \quad , y = t^2 \qquad \frac{dz}{dt} = ?$

$\frac{dz}{dx} = e^x \sin y$

$\frac{dz}{dy} = e^x \cos y$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 2t$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

$$e^x \left(\frac{\sin y}{t} + \cos y \, 2t \right)$$

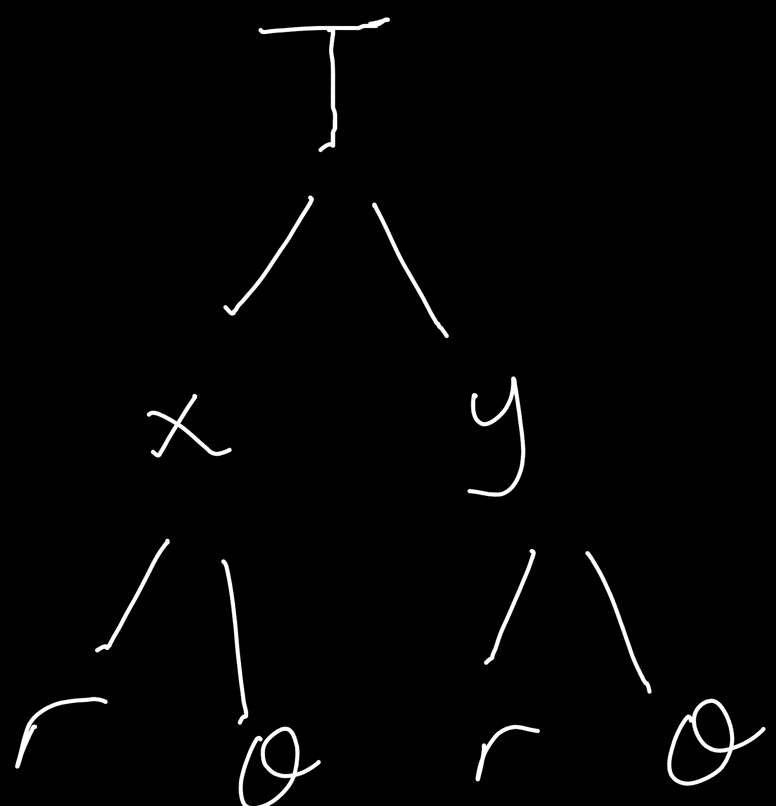
$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) 2t \right.$$

$$\sin(t^2) + 2t^2 \cos(t^2)$$

$T = x^3 - xy + y^3 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$\frac{dT}{dr} = ?$

$\frac{dT}{d\theta} = ?$



$$\frac{dT}{dr} = \frac{dT}{dx}\cdot\frac{dx}{dr} + \frac{dT}{dy}\cdot\frac{dy}{dr}$$

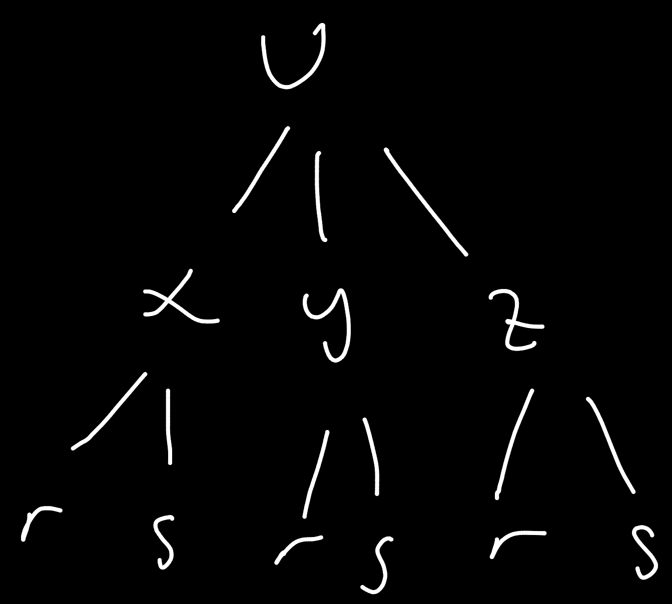
$$\frac{dT}{d\theta} = \frac{dT}{dx}\cdot\frac{dx}{d\theta} + \frac{dT}{dy}\cdot\frac{dy}{d\theta}$$

$$\frac{dT}{dr} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{dT}{d\theta} = (3x^2 - y)\cdot -r\sin\theta + (3y^2 - x)\,r\cos\theta$$

$u = z\sin\left(\frac{y}{x}\right), \quad x = 3r^2 + s, \quad y = 4r - 2s^3, \quad z = 2r^2 - 3s^2$

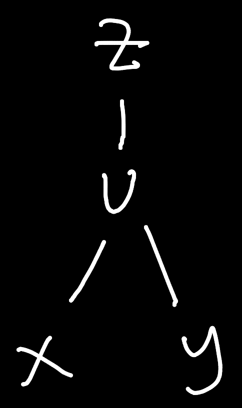
$\frac{du}{ds} = ?, \quad \frac{du}{dr} = ?$



$$\frac{du}{ds} = \frac{du}{dx}\cdot\frac{dx}{ds} + \frac{du}{dy}\cdot\frac{dy}{ds} + \frac{du}{dz}\cdot\frac{dz}{ds}$$

$$\frac{du}{dr} = \frac{du}{dx}\cdot\frac{dx}{dr} + \frac{du}{dy}\cdot\frac{dy}{dr} + \frac{du}{dz}\cdot\frac{dz}{dr}$$

$z = f(u) \quad u = x^2 y$ prove that $x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{1}{2xy}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial u} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$\frac{1}{2xy}\frac{\partial z}{\partial x} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$$