$$f(x,y) = 2x^3 + 3xy^2$$

$$\boxed{F_{xy} = \frac{d^2 f}{dy\,dx}}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$\frac{d^2 f}{dx^2} = 12x$$

$$\frac{d^3 f}{dx^3} = 12$$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\frac{df}{dy}$$

$$\frac{df}{dy} = 6xy$$

$$6xy\frac{d}{dx} = 6y$$

$$\frac{df}{dx} = 6xy$$

$$\frac{d^2 f}{dx^2} = 6x$$

$$\frac{d^3 f}{dy^3} = 0$$

$$\frac{d^2 f}{dy\,dx} = \frac{d}{dy}\frac{df}{dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$6x^2 + 3y^2\frac{d}{dy} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$u = (x^2+y^2+z^2)^{-\frac{1}{2}} \Rightarrow \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$$

$$\frac{du}{dx} = 2x \cdot \frac{-1}{2} (x^2+y^2+z^2)^{-\frac{3}{2}} = -x (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dx^2} = -(x^2+y^2+z^2)^{-\frac{3}{2}} - x \cdot 2x \left(-\frac{3}{2}\right) (x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dy} = 2y \cdot \frac{-1}{2} (x^2+y^2+z^2)^{-\frac{3}{2}} = -y (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dy^2} = -(x^2+y^2+z^2)^{-\frac{3}{2}} - y \cdot 2y \left(-\frac{3}{2}\right) (x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{du}{dz} = 2z \cdot -\frac{1}{2} (x^2+y^2+z^2)^{-\frac{3}{2}} = -z \cdot (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{d^2u}{dz^2} = -(x^2+y^2+z^2)^{-\frac{3}{2}} - z \cdot 2z \cdot \left(-\frac{3}{2}\right) (x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \frac{d^2 z}{d_x dy}\Big|_{(1,1)} = ? \qquad \frac{d}{dx}\left(\frac{dz}{dy}\right) = z_{yx}$$

$$\frac{dz}{dy} = x^2\left(\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) = \frac{x.x^2}{x^2+y^2} = \frac{x^3}{x^2+y^2}$$

$$\frac{d}{dx}\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2)-2x^4}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$\frac{d^2 z}{dx dy}\Big|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y

$f(x, y)$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$\Delta x \to 0 \quad \Delta y \to 0$

$\Delta x \cong dx \quad \Delta y \cong dy$

$$\Delta z = \frac{df}{dx} \cdot \Delta x + \frac{df}{dy} \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$\Delta x \to 0 \quad \Delta y \to 0$

$$\Delta z = \underbrace{\frac{df}{dx} dx + \frac{df}{dy} dy}_{\text{Important Part = Diffentiation of } f} + \varepsilon_1 dx + \varepsilon_2 dy$$

$z = f(x,y) = x^2y - 3y \Rightarrow \Delta z = ? \quad dz = ?$

$\Delta x = -0,01 \qquad \Delta y = 0,02$

$$\Delta z = \left[(x+\Delta x)^2(y+\Delta y) - 3(y+\Delta y)\right] - (x^2y - 3y)$$

$$\left[(x^2+2x\Delta x+(\Delta x)^2)(y+\Delta y) - 3(y+\Delta y)\right] - y(x^2-3)$$

$$y+\Delta y\left(x^2+2x\Delta x+(\Delta x)^2-3\right)$$

$$\cancel{yx^2} + y2x\Delta x + y(\Delta x)^2 \cancel{-3y} + \Delta y x^2 + \Delta y\, 2x\Delta x + \Delta y(\Delta x)^2 - 3\Delta y \cancel{-yx^2} \cancel{+3y}$$

$$\underbrace{2xy\Delta x + x^2\Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x\Delta y + (\Delta x^2)y + \Delta x^2\Delta y}_{\text{ignored}}$$

$$\Delta z = 2xy\Delta x + (x^2-3)\Delta y$$

$$\cong$$

$$dz = 2xy\,dx + x^2 - 3\,dy$$

As we seen here when Δx and Δy go to 0 then $dz \cong \Delta z$

$u = x^2 e^{y/x} \Rightarrow du = ?$

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy \ldots\ldots$$

$$dx = \left(2x e^{y/x} + \frac{y}{x^2} e^{y/x} x^2\right) dx$$

$$dy = (x e^{y/x}) dy$$

$$du = e^{y/x}(2x+y)\,dx + e^{y/x} x \, dy$$

Chain Rule

$z = f(x,y) \quad x = g(r,s) \quad y = h(r,s)$

$$\begin{tikzcd}[row sep=small] & x \arrow[r] \arrow[dr] & r \\ z \arrow[ur] \arrow[dr] & & s \\ & y \arrow[r] \arrow[dr] & r \\ & & s \end{tikzcd}$$

$$\frac{df}{dr} = \frac{df}{dx}\cdot\frac{dx}{dr} + \frac{df}{dy}\cdot\frac{dy}{dr}$$

$$\frac{df}{ds} = \frac{df}{dx}\cdot\frac{dx}{ds} + \frac{df}{dy}\cdot\frac{dy}{ds}$$

$z = e^x \sin y$, $x = \ln t$, $y = t^2$ $\quad \frac{dz}{dt} = ?$

$\frac{dz}{dx} = e^x \sin y$

$\frac{dz}{dy} = e^x \cos y$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 2t$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

$$e^x \left(\frac{\sin y}{t} + \cos y \, 2t \right)$$

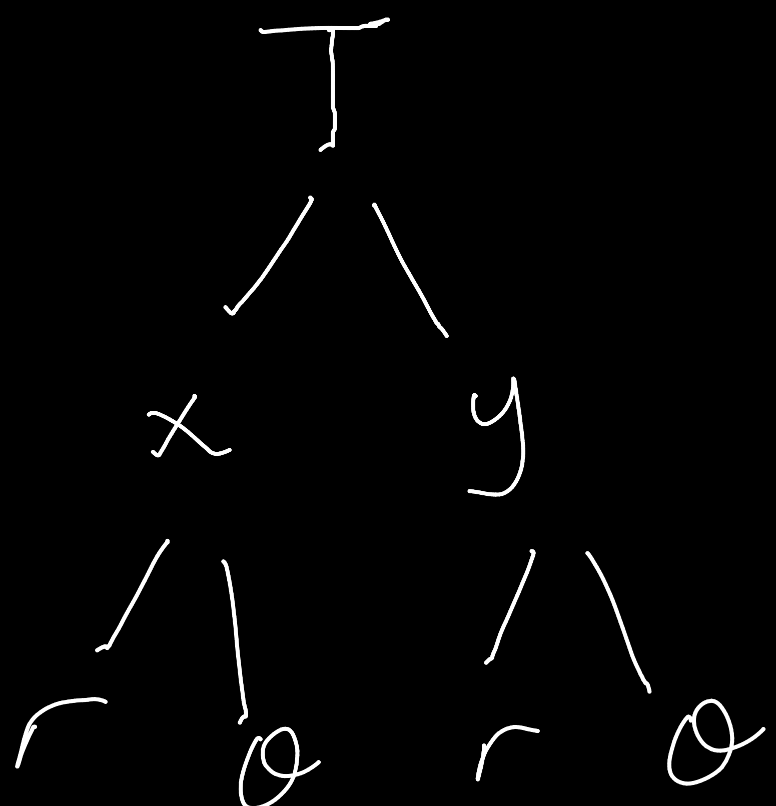
$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) 2t \right.$$

$$\sin(t^2) + 2t^2 \cos(t^2)$$

$T = x^3 - xy + y^3 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$$\frac{dT}{dr} = ?$$

$$\frac{dT}{d\theta} = ?$$



$$\frac{dT}{dr} = \frac{dT}{dx}\cdot\frac{dx}{dr} + \frac{dT}{dy}\cdot\frac{dy}{dr}$$

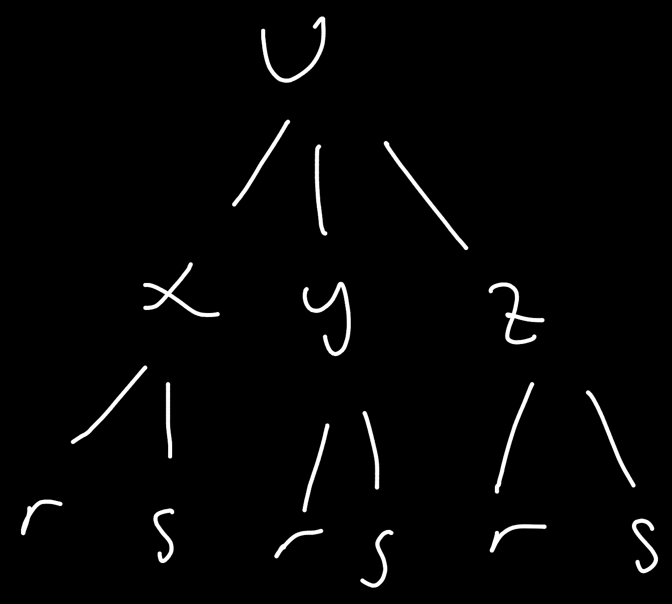
$$\frac{dT}{d\theta} = \frac{dT}{dx}\cdot\frac{dx}{d\theta} + \frac{dT}{dy}\cdot\frac{dy}{d\theta}$$

$$\frac{dT}{dr} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{dT}{d\theta} = (3x^2 - y)\cdot -r\sin\theta + (3y^2 - x)\,r\cos\theta$$

$u = z\sin\left(\frac{y}{x}\right)$, $x = 3r^2 + s$, $y = 4r - 2s^3$, $z = 2r^2 - 3s^2$

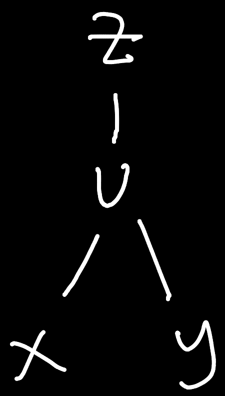
$\frac{du}{ds} = ?$, $\frac{du}{dr} = ?$



$$\frac{du}{ds} = \frac{du}{dx}\cdot\frac{dx}{ds} + \frac{du}{dy}\cdot\frac{dy}{ds} + \frac{du}{dz}\cdot\frac{dz}{ds}$$

$$\frac{du}{dr} = \frac{du}{dx}\cdot\frac{dx}{dr} + \frac{du}{dy}\cdot\frac{dy}{dr} + \frac{du}{dz}\cdot\frac{dz}{dr}$$

$z = f(u)$ $\quad u = x^2 y$ $\quad$ prove that $\quad x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{1}{2xy}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial u} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$\frac{1}{2xy}\frac{\partial z}{\partial x} = \frac{1}{x^2}\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$$

$x^5 + y = t \qquad u = x^3 y \qquad \frac{du}{dt} = ?$

$x^2 + y^3 = t^2$

$$
\begin{tikzcd}
 & u \arrow[dl, no head] \arrow[dr, no head] & \\
x \arrow[d, no head] & & y \arrow[d, no head] \\
t & & t
\end{tikzcd}
$$

$$\frac{du}{dt} = \underbrace{\frac{du}{dx}}_{3x^2y} \cdot \frac{dx}{dt} + \underbrace{\frac{du}{dy}}_{x^3} \cdot \frac{dy}{dt}$$

$$\frac{d}{dt}\left(x^5 + y = t\right) = 5x^4\frac{dx}{dt} + \frac{dy}{dt} = 1$$

$$\frac{d}{dt}\left(x^2 + y^3 = t^2\right) = 2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 2t$$

$$\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$\frac{dx}{dt} = \frac{\begin{bmatrix} 1 & 1 \\ 2t & 3y^2 \end{bmatrix}}{\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix}} = \frac{3y^2 - 2t}{15x^4y^2 - 2x}$$

$$\frac{dy}{dt} = \frac{\begin{vmatrix} 5x^4 & 1 \\ 2x & 2t \end{vmatrix}}{\begin{vmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{vmatrix}} = \frac{10x^4t - 2x}{15x^4y^2 - 2x}$$

$$\frac{du}{dt} = 3x^2y\left(\frac{3y^2 - 2t}{15x^4y^2 - 2x}\right) + x^3\left(\frac{10x^4t - 2x}{15x^4y^2 - 2x}\right)$$

Jacobian

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$F(u,v,w)=0 \quad G(u,v,w)=0 \quad H(u,v,w)=0$

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$F(x,y,u,v)=0 \quad G(x,y,u,v)=0$

$$\frac{du}{dx} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dx} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{du}{dy} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{dv}{dy} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$u^2 - v = 3x + y \qquad \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = ?$$

$$u - 2v^2 = x - 2y$$

$$f(u, v, x, y) = 0 \Rightarrow u^2 - v - 3x - y = 0$$

$$g(u, v, x, y) = 0 \Rightarrow u - 2v^2 - x + 2y = 0$$

$$\frac{\partial u}{\partial x} = - \frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} -3 & -1 \\ -1 & -4v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{12v - 1}{-8u + 1}$$

$$\frac{\partial u}{\partial y} = - \frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} -1 & -1 \\ 2 & -4v \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{4v + 2}{-8u + 1}$$

$$\frac{\partial v}{\partial x} = - \frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} 2u & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{-2u + 3}{-8v + 1}$$

$$\frac{\partial v}{\partial y} = - \frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{\begin{vmatrix} 2u & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{4u + 1}{-8u + 1}$$

$z^3 - xz - y = 0 \qquad \frac{d^2 z}{dx\,dy} = ?$

It is closed-form but we have 2 independent but 1 dependent variable that's why we can't use Jacobian

$$\frac{dz}{dx\,dy} = \frac{d}{dx}\left(\frac{dz}{dy}\right)$$

$$\frac{dz}{dy} = 3z^2\frac{dz}{dy} - x\frac{dz}{dy} - 1 = 0 \qquad \frac{dz}{dy} = \frac{1}{3z^2 - x}$$

we still don't know

$$\frac{d}{dx}\left(\frac{1}{3z^2 - x}\right) = \frac{d(3z^2-x)^{-1}}{dx} = -(3z^2-x)^{-2}\cdot\left(6z\frac{dz}{dx} - 1\right) = \frac{6z\frac{dz}{dx} - 1}{-(3z^2-x)^2}$$

$$\frac{dz}{dx} = 3z^2\frac{dz}{dx} - z - x\frac{dz}{dx} = 0 = \frac{dz}{dx} = \frac{z}{(3z^2-x)}$$

$$\frac{\frac{6z^2}{3z^2-x} - 1}{-(3z^2-x)^2}$$

$u = \sqrt{xy}$

$v = e^{-xy} + xy$

is there a connection between them?

If it's then how?

$$\begin{array}{ccc} & u & \\ / & & \backslash \\ x & & y \end{array} \qquad \begin{array}{ccc} & v & \\ / & & \backslash \\ x & & y \end{array}$$

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \dfrac{y}{2\sqrt{xy}} & \dfrac{x}{2\sqrt{xy}} \\ -y e^{-xy} + y & -x e^{-xy} + X \end{vmatrix} = 0$$

so there is a functional connection between u and v

$u^2 = xy$

$v = e^{-xy} + xy \rightarrow v - e^{-u^2} + u^2 = 0$