

$$f(x, y) = 2x^3 + 3xy^2$$

$$F_{xy} = \frac{d^2 f}{dy dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$\frac{d^2 f}{dx^2} = 12x$$

$$\frac{d^3 f}{dx^3} = 12$$

$$\frac{d^2 f}{dx dy} = \frac{d}{dx} \frac{df}{dy}$$

$$\frac{df}{dy} = 6xy$$

$$6xy \frac{d}{dx} = 6y$$

$$\frac{df}{dx} = 6xy$$

$$\frac{d^2 f}{dx^2} = 6x$$

$$\frac{d^3 f}{dy^3} = 0$$

$$\frac{d^2 f}{dy dx} = \frac{d}{dy} \frac{df}{dx}$$

$$\frac{df}{dx} = 6x^2 + 3y^2$$

$$(6x^2 + 3y^2) \frac{d}{dy} = 6y$$

F_{xy} and F_{yx} don't need to be equal. If there are partial derivatives for every xy then $F_{xy} = F_{yx}$

Laplace

$$v = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$\frac{\partial v}{\partial x} = 2x \cdot \frac{-1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} = -x (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial x^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - x \cdot 2x \left(-\frac{3}{2}\right) (x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$\frac{\partial v}{\partial y} = 2y \cdot \frac{-1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} = -y (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial y^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - y \cdot 2y \left(-\frac{3}{2}\right) (x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$\frac{\partial v}{\partial z} = 2z \cdot \frac{-1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} = -z (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 v}{\partial z^2} = -(x^2 + y^2 + z^2)^{-\frac{3}{2}} - z \cdot 2z \cdot \left(-\frac{3}{2}\right) (x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

$$z = x^2 \arctan\left(\frac{y}{x}\right) \Rightarrow \frac{\partial^2 z}{\partial x \partial y} \Big|_{(1,1)} = ? \quad \frac{d}{dx} \left(\frac{dz}{dy} \right) = z_{yx}$$

$$\frac{dz}{dy} = x^2 \left(\frac{\frac{1}{x}}{1 + \left(\frac{y}{x}\right)^2} \right) = \frac{x \cdot x^2}{x^2 + y^2} = \frac{x^3}{x^2 + y^2}$$

$$\frac{d}{dx} \left(\frac{x^3}{x^2 + y^2} \right) = \frac{3x^2(x^2 + y^2) - 2x^4}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2 y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} \Big|_{(1,1)} = \frac{1+3}{(1+1)^2} = 1$$

Let's assign Δx as changes for x and Δy for y
 $f(x, y)$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$\Delta x \rightarrow 0 \quad \Delta y \rightarrow 0$$

$$\Delta x \approx dx \quad \Delta y \approx dy$$

$$\Delta z = \frac{df}{dx} \cdot \Delta x + \frac{df}{dy} \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 is partial derivative for Δx and ε_2 for Δy

$$\Delta x \rightarrow 0 \quad \Delta y \rightarrow 0$$

$$\Delta z = \frac{df}{dx} dx + \frac{df}{dy} dy + \varepsilon_1 dx + \varepsilon_2 dy$$

[

Important part = Differentiation of f

$$z = f(x, y) = x^2 y - 3y \Rightarrow \Delta z = ? \quad dz = ?$$

$$\Delta x = -0.01 \quad \Delta y = 0.02$$

$$\Delta z = [(x + \Delta x)^2 (y + \Delta y) - 3(y + \Delta y)] - (x^2 y - 3y)$$

$$[(x^2 + 2x\Delta x + (\Delta x)^2)(y + \Delta y) - 3(y + \Delta y)] - y(x^2 - 3)$$

$$y + \Delta y (x^2 + 2x\Delta x + (\Delta x)^2 - 3)$$

$$\cancel{yx^2} + \cancel{y2x\Delta x} + \cancel{y(\Delta x)^2} - \cancel{3y} + \Delta y x^2 + \Delta y 2x\Delta x + \Delta y (\Delta x)^2 - \cancel{3\Delta y} - \cancel{yx^2} + \cancel{3y}$$

$$\underbrace{2xy\Delta x + x^2\Delta y - 3\Delta y}_{\text{important}} + \underbrace{2x\Delta x\Delta y + (\Delta x^2)y + \Delta x^2\Delta y}_{\text{ignored}}$$

important

ignored

$$\Delta z = 2xy\Delta x + (x^2 - 3)\Delta y$$

$$dz = 2xydx + (x^2 - 3)dy \quad \approx$$

As we seen here when Δx and Δy

go to 0 then $dz \approx \Delta z$

$$v = x^2 e^{y/x} \Rightarrow dv =$$

$$dv = \frac{dv}{dx} dx + \frac{dv}{dy} dy \dots$$

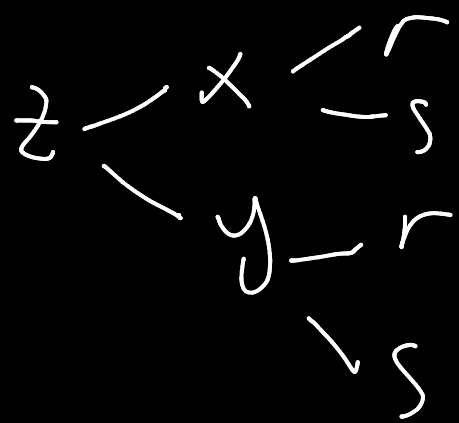
$$dx = \left(2x e^{y/x} + \frac{y}{x^2} e^{y/x} x^2 \right) dx$$

$$dy = (x e^{y/x}) dy$$

$$dv = e^{y/x} (2x + y) dx + e^{y/x} dy$$

Chain Rule

$$z = f(x, y) \quad x = g(r, s) \quad y = h(r, s)$$



$$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr}$$

$$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}$$

$$z = e^x \sin y, \quad x = \ln t, \quad y = t^2 \quad \frac{dz}{dt} = ?$$

$$\frac{dz}{dx} = e^x \sin y$$

$$\frac{dz}{dy} = e^x \cos y$$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\frac{dy}{dt} = 2t$$

$$e^x \sin y \cdot \frac{1}{t} + e^x \cos y \cdot 2t$$

$$e^x \left(\frac{\sin y}{t} + \cos y \cdot 2t \right)$$

$$e^{\ln t} \left(\frac{\sin(t^2)}{t} + \cos(t^2) \cdot 2t \right)$$

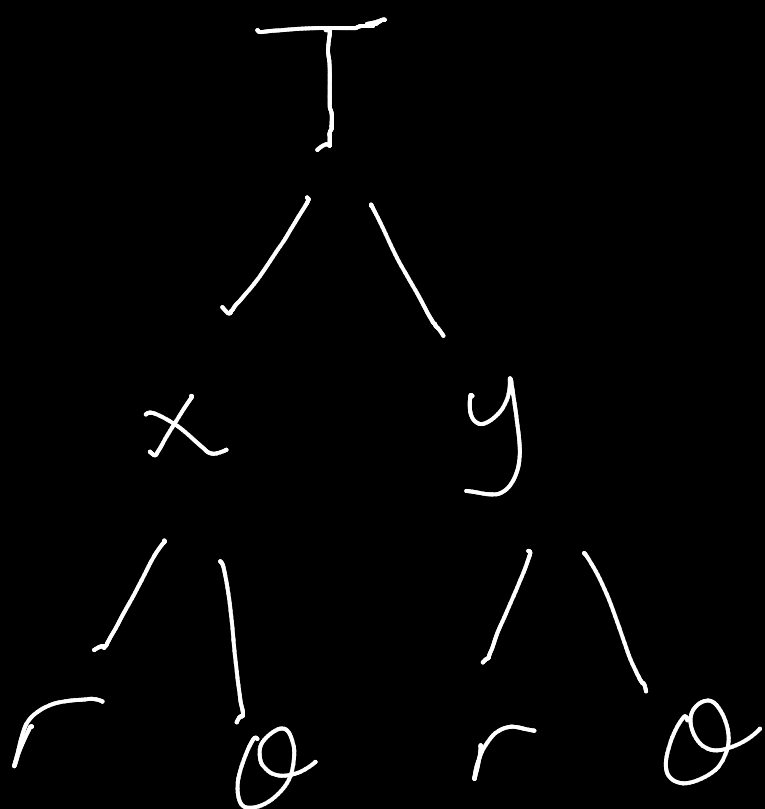
$$\sin(t^2) + 2t^2 \cos(t^2)$$

$$T = x^3 - xy + y^3$$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$\frac{dT}{dr} = ?$$

$$\frac{dT}{d\theta} = ?$$



$$\frac{dT}{dr} = \frac{dT}{dx} \cdot \frac{dx}{dr} + \frac{dT}{dy} \cdot \frac{dy}{dr}$$

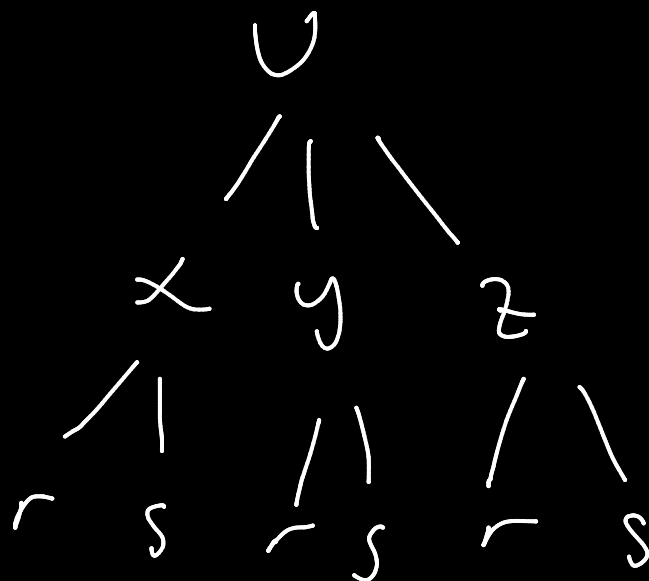
$$\frac{dT}{d\theta} = \frac{dT}{dx} \cdot \frac{dx}{d\theta} + \frac{dT}{dy} \cdot \frac{dy}{d\theta}$$

$$\frac{dT}{dr} = (3x^2 - y) \cos \theta + (3y^2 - x) \sin \theta$$

$$\frac{dT}{d\theta} = (3x^2 - y) \cdot -r \sin \theta + (3y^2 - x) r \cos \theta$$

$$v = z \sin\left(\frac{y}{x}\right), \quad x = 3r^2 + s, \quad y = 4r - 2s^3, \quad z = 2r^2 - 3s^2$$

$$\frac{dv}{ds} = ? \quad \frac{dv}{dr} = ?$$



$$\frac{dv}{ds} = \frac{dv}{dx} \cdot \frac{dx}{ds} + \frac{dv}{dy} \cdot \frac{dy}{ds} + \frac{dv}{dz} \cdot \frac{dz}{ds}$$

$$\frac{dv}{dr} = \frac{dv}{dx} \cdot \frac{dx}{dr} + \frac{dv}{dy} \cdot \frac{dy}{dr} + \frac{dv}{dz} \cdot \frac{dz}{dr}$$

$$z = f(u) \quad u = x^2 y$$

prove that

$$x \frac{\partial z}{\partial x} = 2y \frac{\partial z}{\partial y}$$

$$\begin{array}{c} z \\ | \\ u \\ / \quad \backslash \\ x \quad y \end{array} \quad \begin{array}{l} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} \\ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} \end{array}$$

$$\left. \begin{array}{l} \frac{\partial z}{\partial u} = \frac{1}{2xy} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial u} = \frac{1}{x^2} \frac{\partial z}{\partial y} \end{array} \right\}$$

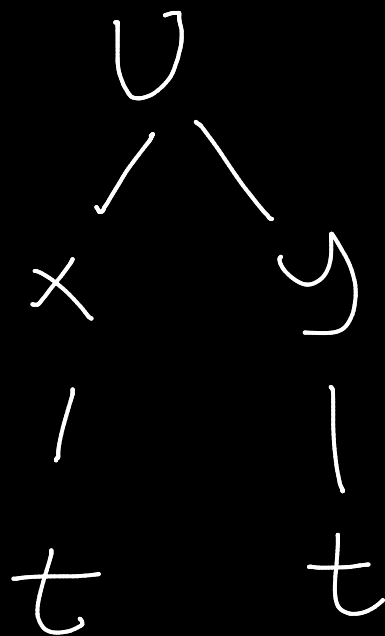
$$\frac{1}{2xy} \frac{\partial z}{\partial x} = \frac{1}{x^2} \frac{\partial z}{\partial y}$$

$$x \frac{\partial z}{\partial x} = 2y \frac{\partial z}{\partial y}$$

$$x^5 + y = t$$

$$v = x^3 y$$

$$\frac{dv}{dt} = ?$$



$$\frac{dv}{dt} = \underbrace{\frac{dv}{dx}}_{3x^2 y} \cdot \frac{dx}{dt} + \underbrace{\frac{dv}{dy}}_{x^3} \cdot \frac{dy}{dt}$$

$$\frac{d}{dt}(x^5 + y = t) = 5x^4 \frac{dx}{dt} + \frac{dy}{dt} = 1$$

$$\frac{d}{dt}(x^2 + y^3 = t^2) = 2x \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 2t$$

$$\begin{bmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$\frac{dx}{dt} = \frac{\begin{vmatrix} 1 & 1 \\ 2t & 3y^2 \end{vmatrix}}{\begin{vmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{vmatrix}} = \frac{3y^2 - 2t}{15x^4 y^2 - 2x}$$

$$\frac{dy}{dt} = \frac{\begin{vmatrix} 5x^4 & 1 \\ 2x & 2t \end{vmatrix}}{\begin{vmatrix} 5x^4 & 1 \\ 2x & 3y^2 \end{vmatrix}} = \frac{10x^4 t - 2x}{15x^4 y^2 - 2x}$$

$$\frac{dv}{dt} = 3x^2 y \left(\frac{3y^2 - 2t}{15x^4 y^2 - 2x} \right) + x^3 \left(\frac{10x^4 t - 2x}{15x^4 y^2 - 2x} \right)$$

Jacobian

$$\frac{\partial(F, G)}{\partial(u, v)} \rightarrow \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$F(u, v, w) = 0 \quad G(u, v, w) = 0 \quad H(u, v, w) = 0$$

$$\frac{\partial(F, G, H)}{\partial(u, v, w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$$F(x, y, u, v) = 0 \quad G(x, y, u, v) = 0$$

$$\frac{du}{dx} = - \frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{dv}{dx} = - \frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{du}{dy} = - \frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\frac{dv}{dy} = - \frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$\begin{aligned} v^2 - u &= 3x + y \\ v - 2u^2 &= x - 2y \end{aligned} \quad \frac{dv}{dx}, \frac{dv}{dy}, \frac{du}{dx}, \frac{du}{dy} = \left\{ \right.$$

$$f(v, u, x, y) = 0 \Rightarrow v^2 - u - 3x - y = 0$$

$$g(v, u, x, y) = 0 \Rightarrow v - 2u^2 - x + 2y = 0$$

$$\frac{dv}{dx} = - \frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ G_v & G_u \end{vmatrix}} = - \frac{\begin{vmatrix} -3 & -1 \\ -1 & -4u \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{12u - 1}{-8v + 1}$$

$$\frac{dv}{dy} = - \frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ G_v & G_u \end{vmatrix}} = - \frac{\begin{vmatrix} -1 & -1 \\ 2 & -4u \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{4u + 2}{-8v + 1}$$

$$\frac{du}{dx} = - \frac{\begin{vmatrix} F_v & F_x \\ G_v & G_x \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ G_v & G_u \end{vmatrix}} = - \frac{\begin{vmatrix} 2v & -3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{-2v + 3}{-8v + 1}$$

$$\frac{du}{dy} = - \frac{\begin{vmatrix} F_v & F_y \\ G_v & G_y \end{vmatrix}}{\begin{vmatrix} F_v & F_u \\ G_v & G_u \end{vmatrix}} = - \frac{\begin{vmatrix} 2v & -1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2v & -1 \\ 1 & -4 \end{vmatrix}} = - \frac{4v + 1}{-8v + 1}$$

$$z^3 - xz - y = 0 \quad \frac{d^2 z}{dx dy} = ?$$

It is closed-form
but we have 2 independent but
1 dependent variable that's why
we can't use jacobian

$$\frac{dz}{dx dy} = \frac{d}{dx} \left(\frac{dz}{dy} \right)$$

$$\frac{dz}{dy} = 3z^2 \frac{dz}{dy} - x \frac{dz}{dy} - 1 = 0$$

$$\frac{dz}{dy} = \frac{1}{3z^2 - x}$$

We still
don't know
↑

$$\frac{d}{dx} \left(\frac{1}{3z^2 - x} \right) = \frac{d(3z^2 - x)^{-1}}{dx} = - (3z^2 - x)^{-2} \cdot \left(6z \frac{dz}{dx} - 1 \right) = \frac{6z \frac{dz}{dx} - 1}{-(3z^2 - x)^2}$$

$$\frac{dz}{dx} = 3z^2 \frac{dz}{dx} - z - x \frac{dz}{dx} = 0 \Rightarrow \frac{dz}{dx} = \frac{z}{(3z^2 - x)}$$

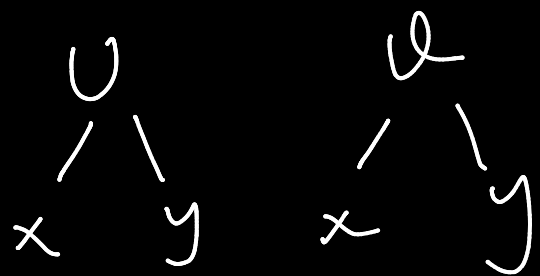
$$\frac{6z^2 - 1}{3z^2 - x} \cdot \frac{1}{-(3z^2 - x)^2}$$

$$v = \sqrt{xy}$$

$$u = e^{-xy} + xy$$

is there a connection
between them?

If it's then how?



$$\begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ -ye^{-xy} + y & -xe^{-xy} + x \end{vmatrix} \neq 0$$

so there is a
functional connection
between v and u

$$v^2 = xy$$

$$u = e^{-xy} + xy \rightarrow u = e^{-v^2} + v^2 = 0$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\theta = 0 \text{ yada } r = 0$$

$$y = r \sin \theta$$

$$\theta = 0 \text{ yada } r = 0$$

Directional

Derivative

$$\frac{dF}{ds} = \nabla F \cdot T$$

$$F = x^2 y z^3$$

$$v = 0$$

$$x = e^{-v}$$

$$y = 2 \sin v + 1$$

$$z = v - \cos v$$

find directional derivative
through x, y, z

$$\nabla F = 2xyz^3 \underline{i} + x^2 z^3 \underline{j} + x^2 y 3z^2 \underline{k}$$

$$T = \frac{T_0}{|T_0|} \quad T_0 = \frac{dr}{dv}$$

$$v=0$$

$$x=1$$

$$y=1$$

$$z=-1$$

$$\nabla F|_{v=0} = -2 \underline{i} - \underline{j} + 3 \underline{k}$$

$$r = x \underline{i} + y \underline{j} + z \underline{k}$$

$$r = e^{-v} \underline{i} + (2 \sin v + 1) \underline{j} + (v - \cos v) \underline{k}$$

$$\frac{dr}{dv} = -e^{-v} \underline{i} + 2 \cos v \underline{j} + (1 + \sin v) \underline{k}$$

$$(-2 \underline{i} - \underline{j} + 3 \underline{k}) \cdot \left(\frac{-\underline{i} + 2 \underline{j} + \underline{k}}{\sqrt{6}} \right)$$

$$T_0|_{v=0} = -\underline{i} + 2 \underline{j} + \underline{k}$$

$$= (2 - 2 + 3) \frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

$$T = \frac{-\underline{i} + 2 \underline{j} + \underline{k}}{\sqrt{6}}$$

$$F = 2x^3y - 3y^2z \quad P(1, 2, -1) \quad Q(3, -1, 5)$$

Find directional derivative of F for P through Q direction.

$$\nabla F \cdot T \quad \nabla F = 6x^2y \underline{i} + (2x^3 - 6yz) \underline{j} - 3y^2 \underline{k}$$

$$\nabla F|_P = 12 \underline{i} + 14 \underline{j} - 12 \underline{k}$$

$$T_0 = r_q - r_p = (3 \underline{i} - \underline{j} + 5 \underline{k}) - (\underline{i} + 2 \underline{j} - \underline{k}) = 2 \underline{i} - 3 \underline{j} + 6 \underline{k}$$

$$\frac{T = T_0}{|T_0|} = \frac{2 \underline{i} - 3 \underline{j} + 6 \underline{k}}{\sqrt{4 + 9 + 36}} = \frac{2 \underline{i} - 3 \underline{j} + 6 \underline{k}}{7}$$

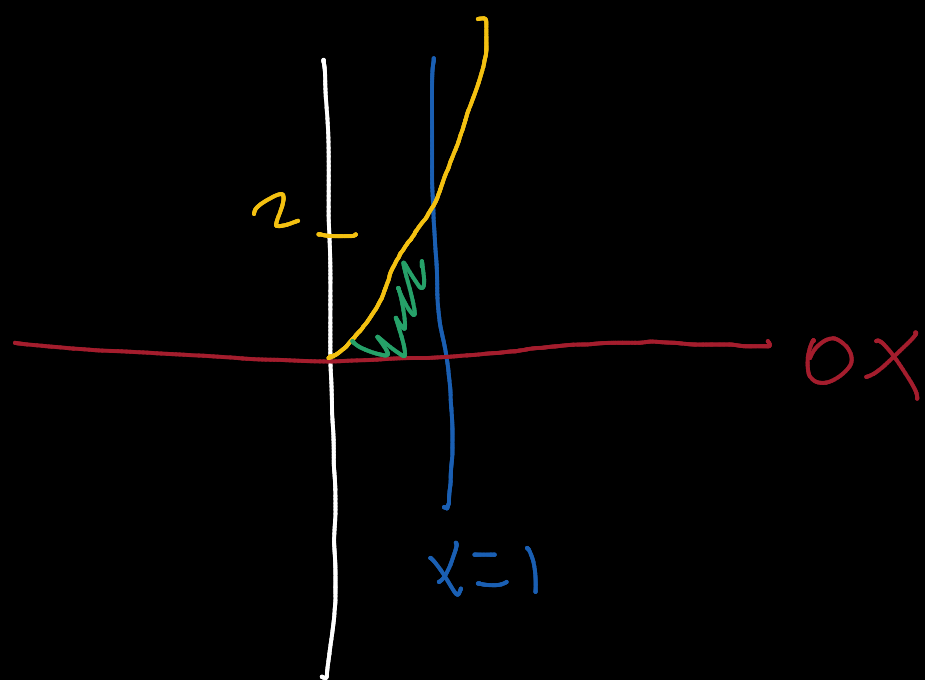
$$\nabla F \cdot T = (12 \underline{i} + 14 \underline{j} - 12 \underline{k}) \cdot \left(\frac{2 \underline{i} - 3 \underline{j} + 6 \underline{k}}{7} \right)$$

$$= \frac{24 - 42 - 72}{7} = \frac{-90}{7}$$

what is the max value of directional derivative? $|\nabla F|_P = \sqrt{12^2 + 14^2 + (-12)^2}$

Double Integral

$$y = 2x^2 \quad x=1 \quad \text{and} \quad 0 \leq x \leq 1 \quad f(x,y) = x^2y - x + y$$



$$\int_0^1 \int_0^{2x^2} (x^2y - x + y) dy dx$$

$$\int_0^1 \left(x^2 \frac{y^2}{2} - xy + \frac{y^2}{2} \right) \bigg|_0^{2x^2} dx$$

$$\frac{x^2(2x^2)^2}{2} - x(2x^2) + \frac{(2x^2)^2}{2}$$

$$\int_0^1 (2x^6 - 2x^3 + 2x^4) dx$$

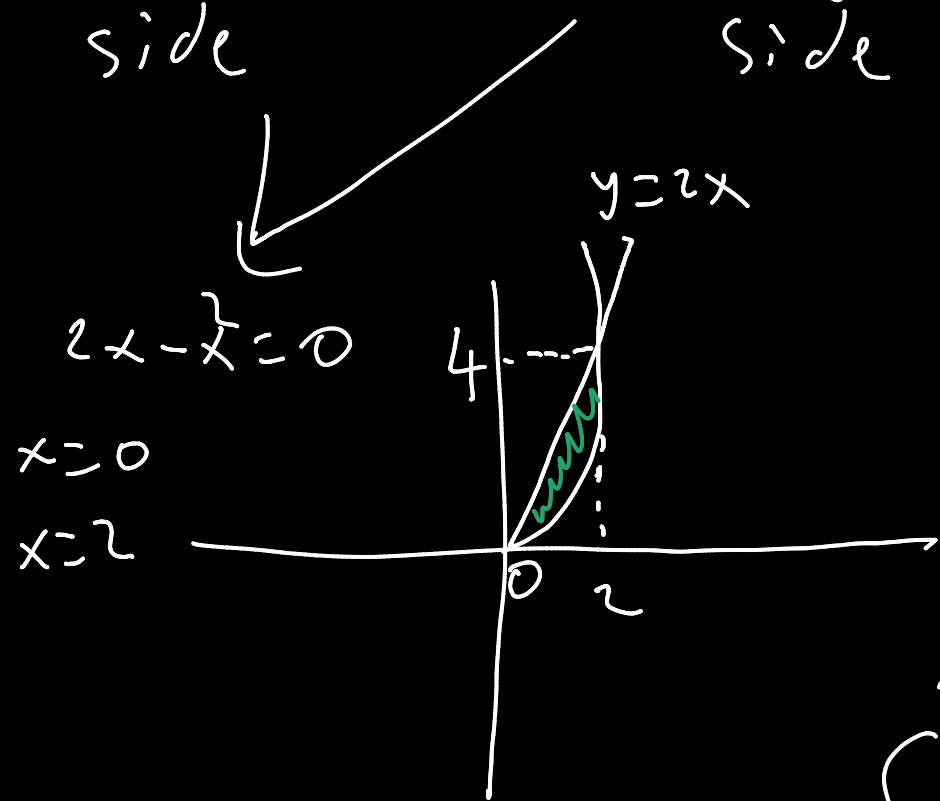
$$2 \left(\frac{x^7}{7} - \frac{x^4}{4} + \frac{x^5}{5} \right) \bigg|_0^1$$

$$2 \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{5} \right) = \frac{20 - 35 + 28}{70} = \frac{13}{70}$$

$$y \leq 2x, \quad x^2 - y \leq 0$$

$$f(x, y) = x^3 + y$$

$y = 2x$ but shrinking side
 $x^2 = y$ but growing side



$$\int_0^2 \int_{x^2}^{2x} (x^3 + y) dy dx$$

$$\int_0^2 \left(x^3 y + \frac{y^2}{2} \right) \Big|_{x^2}^{2x} dx$$

$$\int_0^2 \left(2x^4 + 2x^2 \right) - \left(x^5 + \frac{x^4}{2} \right) dx$$

$$\left(\frac{2x^5}{5} + \frac{2x^3}{3} - \frac{x^6}{6} - \frac{x^5}{10} \right) \Big|_0^2$$

$$\frac{64}{5} + \frac{16}{3} - \frac{64}{6} - \frac{32}{10}$$

$$= \frac{384 + 160 - 320 - 96}{30} = \frac{128}{30}$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3} \approx 1.33$$

$$= \frac{64}{15}$$

Area of the defined place

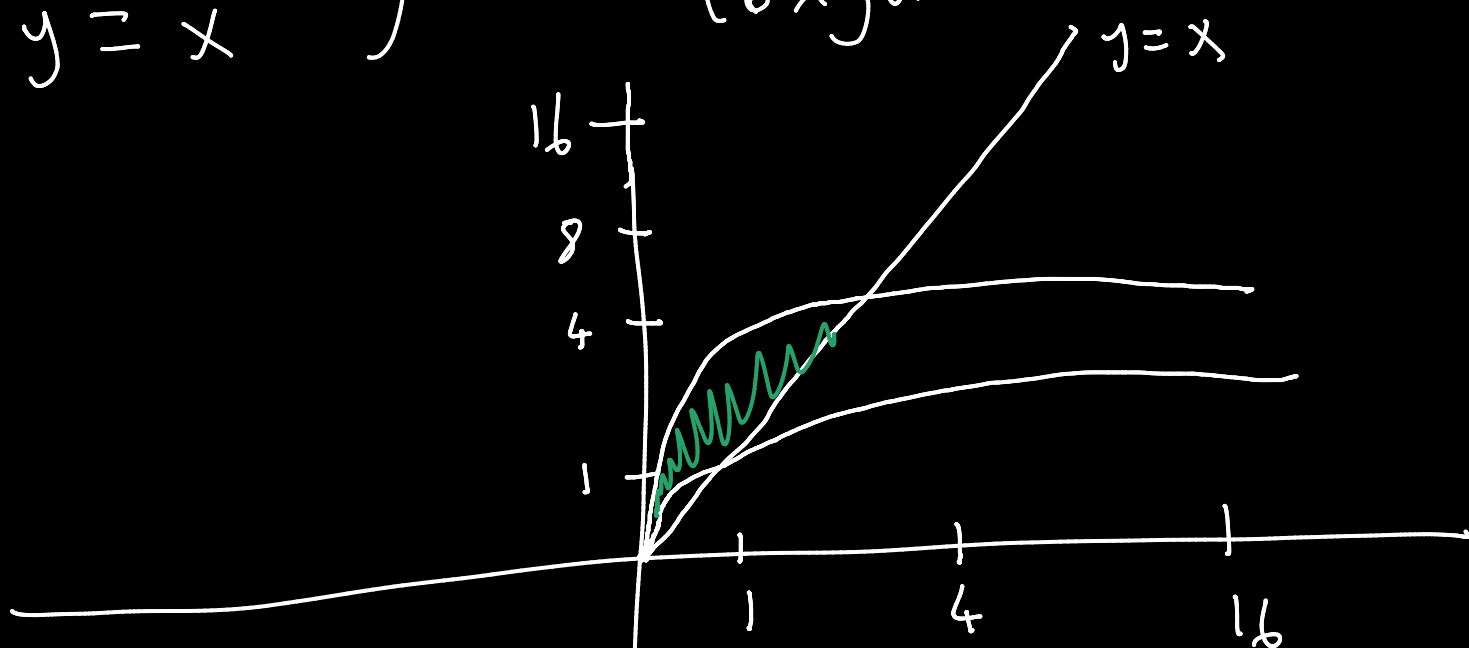
$$\int_0^2 \int_{x^2}^{2x} dy dx$$

$$\int_0^2 (2x - x^2) dx$$

$$\left(x^2 - \frac{x^3}{3} \right) \Big|_0^2$$

$y = 2\sqrt{x}$
 $y = \sqrt{x}$
 $y = x$

Area of the this
conjunction



$$\int_0^1 \int_{\sqrt{x}}^{2\sqrt{x}} dy dx + \int_1^4 \int_x^{2\sqrt{x}} dy dx$$

$$\int_0^1 (2\sqrt{x} - \sqrt{x}) dx + \int_1^4 (2\sqrt{x} - x) dx$$

$$\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 + \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} \Big|_1^4$$

$$= \frac{2}{3} + \left(\frac{32}{3} - 8 \right) - \left(\frac{4}{3} - \frac{1}{2} \right)$$

$$= \frac{2}{3} + \frac{8}{3} - \frac{5}{6} = \frac{4}{6} + \frac{16}{6} - \frac{5}{6} = \frac{15}{6} = \frac{5}{2}$$

$$x^2 + y^2 \leq 4x \quad \text{area of this?}$$

$$x^2 + y^2 - 4x = 0$$

$$x^2 + y^2 - 4x + 4 - 4 = 0$$

$$x^2 - 4x + 4 + y^2 = 4$$

$$(x-2)^2 + y^2 = 4$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$$y = \sqrt{4x-4}$$

$$y = \sqrt{4 - (x-2)^2}$$

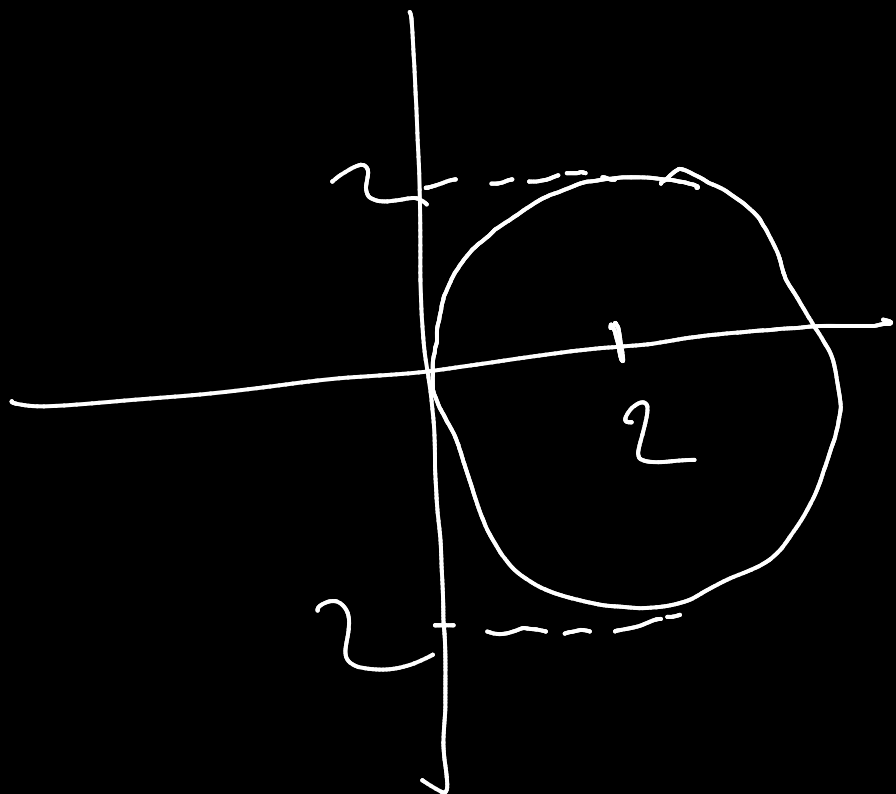
we choose this for easier int's

$$dx = dv$$

$$A = \int_0^2 \int_0^{\sqrt{4-(x-2)^2}} dy dx$$

$$= \int_{-2}^2 \sqrt{4-v^2} dv = \int_{-2}^2 \sqrt{2^2 - v^2} dv$$

$$\sqrt{a^2 - x^2} \rightarrow x = a \sin t$$



$$v = 2 \sin t$$

$$dv = 2 \cos t dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4 - (2 \sin t)^2} \cdot 2 \cos t dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1 - \sin^2 t)} \cdot 2 \cos t dt$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2t) dt$$

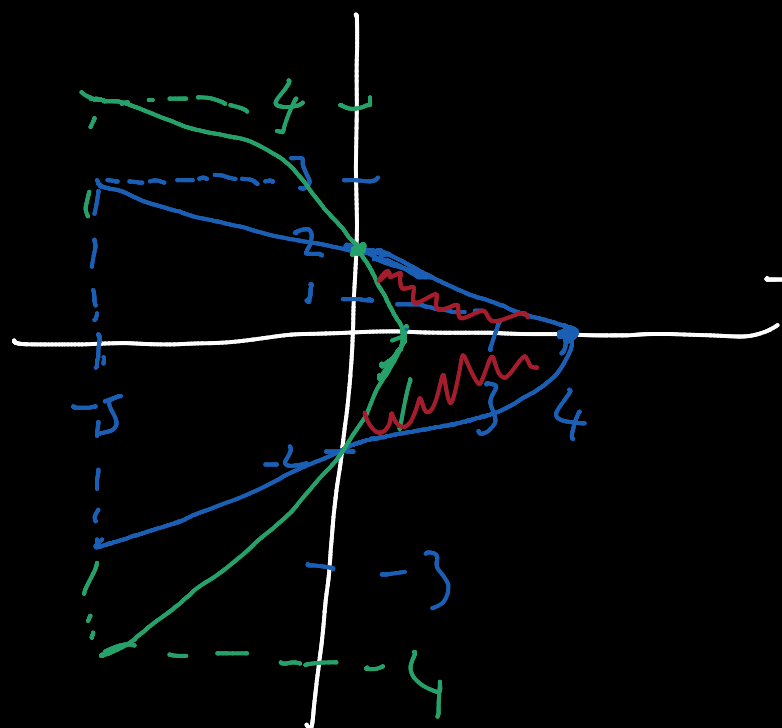
$$2 \left(t + \frac{\sin 2t}{2} \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = 2\pi = \frac{A}{2}$$

$$A = 4\pi$$

$$y^2 = 4 - x$$

$$y^2 = 4 - 4x$$

Area ?



$$x = \frac{4 - y^2}{4} \quad x = 4 - y^2$$

$$\int_{-2}^2 \int_{\frac{4-y^2}{4}}^{4-y^2} dx dy$$

$$= \int_{-2}^2 \left(4 - y^2 - \left(\frac{4 - y^2}{4} \right) \right) dy$$

$$= \frac{1}{4} \int_{-2}^2 (16 - 4y^2 - 4 + y^2) dy$$

$$= \frac{1}{4} \int_{-2}^2 (12 - 3y^2) dy$$

$$= \frac{1}{4} \left(12y - y^3 \right) \Big|_{-2}^2$$

$$= \frac{1}{4} (24 - 8) - (-24 + 8)$$

$$= 8 \text{ Area}$$

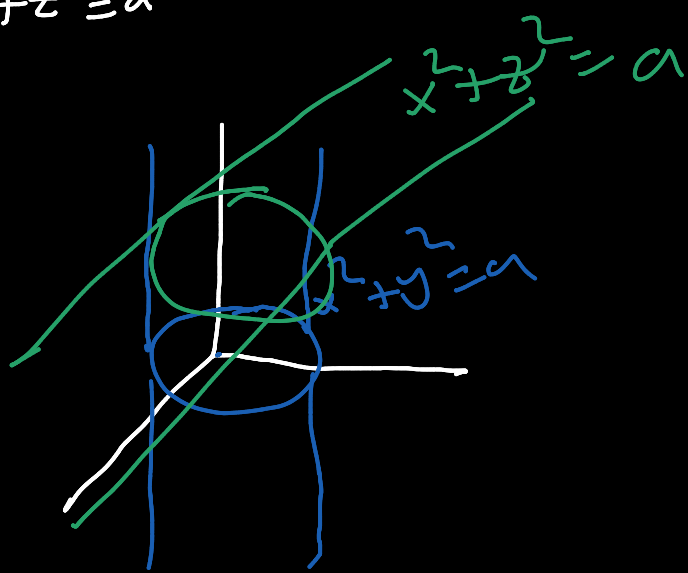
$$x^2 + y^2 = a^2$$

$$x^2 + z^2 = a^2$$

Volume

$$z = \sqrt{a^2 - x^2}$$

$$y = \sqrt{a^2 - x^2}$$



$$V = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} \, dy \, dx$$

$$\int_{-a}^a \left[\sqrt{a^2-x^2} y \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \, dx = 2 \int_{-a}^a (a^2 - x^2) \, dx$$

$$= 2 \left(\left[a^2 x - \frac{x^3}{3} \right]_{-a}^a \right) = 2 \left(\left(a^3 - \frac{a^3}{3} \right) - \left(-a^3 + \frac{a^3}{3} \right) \right)$$

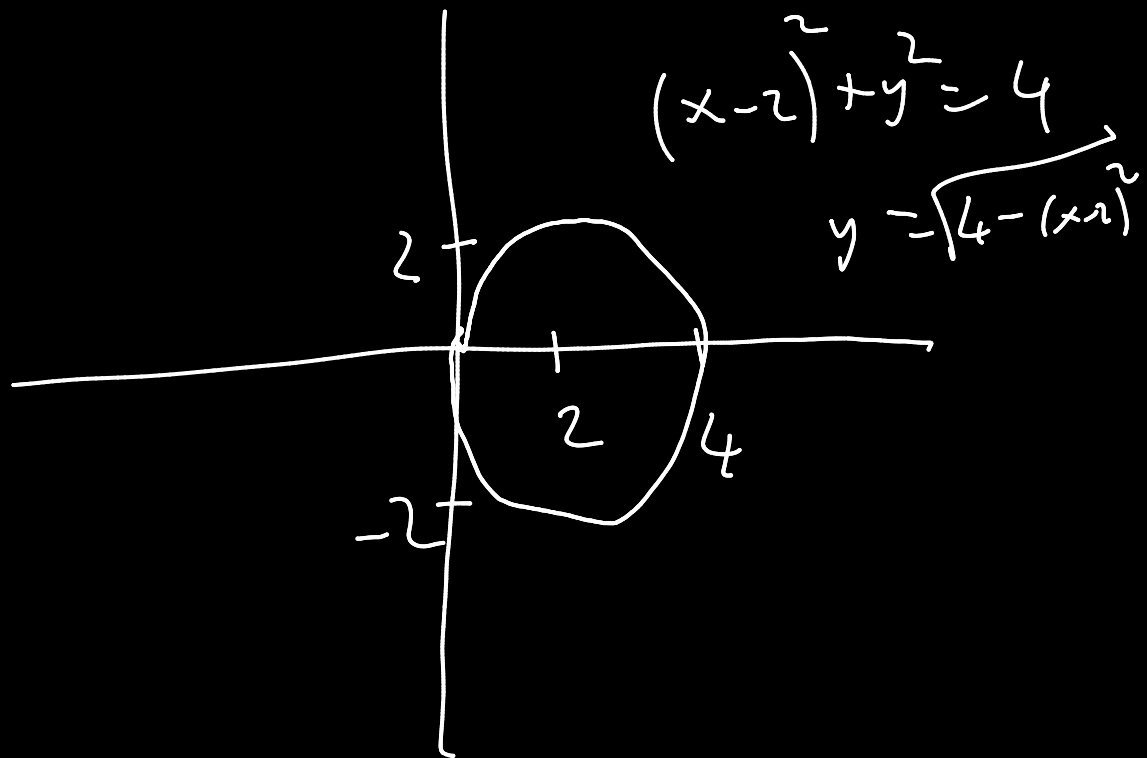
$$= 2 \left(2a^3 - \frac{2a^3}{3} \right) = 4a^3 - \frac{4a^3}{3} = \frac{8a^3}{3}$$

$$z=0 \quad x^2+y^2 \leq 4x \quad z+x=8 \quad \text{Volume!}$$

$$y^2 = 4x - x^2$$

$$y = \pm \sqrt{4x - x^2}$$

$$\int_0^4 \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} (8-x) dy dx$$



$$\int_0^4 (8y - xy) \Big|_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} dx = \int_0^4 (8\sqrt{4-(x-2)^2} - x\sqrt{4-(x-2)^2}) dx$$

$$= \int_0^4 (16\sqrt{4-(x-2)^2} - 2x\sqrt{4-(x-2)^2}) dx = 2 \int_0^4 (8-x)\sqrt{4-(x-2)^2} dx$$

$\sqrt{a^2 - x^2}$
 $x = a \sin t$
 $dx = a \cos t dt$

$$16 \int_0^4 \sqrt{4-(x-2)^2} dx - 2 \int_0^4 x \sqrt{4-(x-2)^2} dx$$

$\downarrow x-2 = 2 \sin t$ $\downarrow x-2 = u$
 $\downarrow dx = 2 \cos t dt$ $\downarrow dx = du$

$$16 \int_{-\pi/2}^{\pi/2} \sqrt{4(1-\sin^2 t)} 2 \cos t dt - 2 \int_{-2}^2 (u+2) \sqrt{4-u^2} du$$

$$64 \int_{-\pi/2}^{\pi/2} \cos^2 t dt - 2 \int_{-2}^2 u \sqrt{4-u^2} du - 4 \int_{-2}^2 \sqrt{4-u^2} du$$

$4-u^2 = p$
 $-2u du = dp$
 $\int p^{1/2} dp = \frac{2p^{3/2}}{3} = \frac{2(4-u^2)^{3/2}}{3}$ $u = 2 \sin m$
 $du = 2 \cos m dm$

$$= \frac{2(4-u^2)^{3/2}}{3} \Big|_{-2}^2 = 0$$

$$-4 \int_{-\pi/2}^{\pi/2} \sqrt{4(1-\sin^2 m)} dm$$

$$= 16 \int_{-\pi/2}^{\pi/2} \cos^2 m dm = -8 \left(m + \cos 2m \right) \Big|_{-\pi/2}^{\pi/2} = -8\pi$$

$$32 \left(t + \frac{\sin 2t}{2} \right) \Big|_{-\pi/2}^{\pi/2}$$

$$= 32 \left[\left(\frac{\pi}{2} + 0 \right) - \left(-\frac{\pi}{2} + 0 \right) \right]$$

$$= 32\pi \quad \rightarrow \quad 24\pi$$

Transformation with Double Integrals

$$x^2 - y^2 = 1$$

$$x^2 - y^2 = 9$$

$$xy = 2$$

$$xy = 4$$

$$x^2 - y^2 = u$$

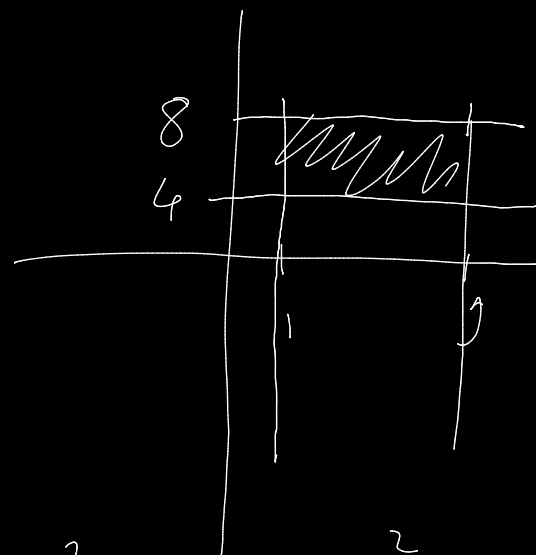
$$u = 1$$

$$u = 9$$

$$2xy = w$$

$$w = 4$$

$$w = 8$$



$$\begin{aligned} u^2 + w^2 &= (x^2)^2 - 2x^2y^2 + (y^2)^2 + 4x^2y^2 \\ &= (x^2 + y^2)^2 \end{aligned}$$

$$x^2 + y^2 = \sqrt{u^2 + w^2}$$

$$J = \frac{2(u, w)}{2(x, y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2$$

$$J = \frac{1}{4x^2 + 4y^2} = \frac{1}{4\sqrt{u^2 + w^2}}$$

$$\int_1^9 \int_4^8 \frac{1}{4\sqrt{u^2 + w^2}} dw du = \frac{1}{4} \int_1^9 4 du = 8$$

