Riccati:

$$R = \frac{dy}{dx} + p(x)y^2 + q(x)y = r(x)$$

Riccati needs at least one private solution to find homogen solution $y = y_1 + \frac{1}{v}$

if $p(x) = 0$ then R will be Linear

else if $r(x) = 0$ then R will be Bernoulli

$$\frac{dv}{dx} - (2y_1 p(x) + q(x))v = p(x) \quad \text{Linear}$$

$$\frac{dy}{dx} + y^2 - (1+2e^x)y = -e^{2x} \qquad y_1 = e^x$$

$p(x) = 1$

$q(x) = -(1+2e^x)$

$r(x) = -e^{2x}$

$$y = e^x + \frac{1}{v}$$

$$\frac{dv}{dx} + (-2e^x + 2e^x + 1)v = 1 = \frac{dv}{dx} + v = 1 \quad (\text{Linear})$$

$$\lambda = e^{\int dx} = e^x$$

$$e^x v = \int e^x dx$$

$$= e^x v = e^x + C$$

$$v = \frac{e^x + C}{e^x}$$

$$y = e^x + \frac{e^x}{e^x + C}$$

$$x^2y' + x^2y^2 + xy = 4 \qquad y_1 = \frac{-2}{x} \qquad y = \frac{-2}{x} + \frac{1}{v}$$

$$y' + y^2 + \frac{1}{x}y = 4$$

$$P(x) = 1$$

$$Q(x) = \frac{1}{x}$$

$$r(x) = 4$$

$$\frac{dv}{dx} - \left(\frac{-4}{x} + \frac{1}{x}\right)v = 1$$

$$= \frac{dv}{dx} + \frac{3}{x}v = -1 \quad \text{(linear)}$$

$$\lambda = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$x^3 v = \int x^3 dx$$

$$x^3 v = \frac{x^4}{4} + C$$

$$v = \frac{x}{4} + C$$

$$y = \frac{-2}{x} + \frac{1}{\frac{x}{4} + C} = \frac{-2 + 4C}{x}$$

$x^2y' + x^2y^2 - 2xy = -2$ $\quad y_1 = \frac{1}{x}$

$y' + y^2 - \frac{2}{x}y = -2$ $\quad y = \frac{1}{x} + \frac{1}{v}$

$p(x) = 1$

$\frac{dv}{dx} - \left(\frac{2}{x} - \frac{2}{x}\right)y = 1$ $\quad Q(x) = -\frac{2}{x}$

$r(x) = -2$

$\frac{dv}{dx} = 1$

$\frac{dy}{dx} - x^3y^2 + \left(2x^4 - \frac{1}{x}\right)y = x^5$ $\quad y_1 = x$

$y = x + \frac{1}{v}$

$\frac{dv}{dx} - \left(-2x^4 + 2x^4 - \frac{1}{x}\right)v = -x^3$ $\quad p(x) = -x^3$

$Q(x) = 2x^4 - \frac{1}{x}$

$\frac{dv}{dx} + \frac{1}{x}v = -x^3$ $\quad r(x) = x^5$

$\int \frac{1}{x}dx$

$\lambda = e = x$

$xv = \int x \cdot -x^3\,dx = \frac{-x^5}{5} + C$ $\quad v = \frac{-x^4}{5} + \frac{C}{x}$ $\quad y = x + \frac{1}{\frac{-x^4}{5} + \frac{C}{x}}$

$$(1-x^3)y' - y^2 + x^2 y = -2x \qquad y_1 = -x^2 \qquad y = -x^2 + \frac{1}{v}$$

$$y' - \frac{1}{(1-x^3)}y^2 + \frac{x^2}{(1-x^3)}y = \frac{-2x}{(1-x^3)}$$

$$P(x) = \frac{-1}{(1-x^3)}$$

$$\frac{dv}{dx} - \left(\frac{2}{1-x^3}x^2 + \frac{x^2}{1-x^3}\right)v = \frac{-1}{(1-x^3)} \qquad Q(x) = \frac{x^2}{(1-x^3)}$$

$$\frac{dv}{dx} - \frac{3x^2}{1-x^3}v = \frac{-1}{(1-x^3)} \qquad \lambda = e^{\int \frac{-3x^2}{1-x^3}dx} \qquad 1-x^3 = v$$

$$-3x^2 dx = dv$$

$$\lambda = 1-x^3$$

$$(1-x^3)v = \int (1-x^3)\frac{-1}{(1-x^3)}dx$$

$$= v = \frac{-x+C}{1-x^3} \qquad y = -x^2 + \frac{1-x^3}{-x+C}$$

Orthogonal Trajectories

$x - 4y = c$

$1 - 4\frac{dy}{dx} = 0$

$1 + 4\frac{dx}{dy} = 0$

$\int dy = -4\int dx$

$y = -4x + c$

$y = ce^{-x} \quad c = \frac{y}{e^{-x}}$

$\frac{dy}{dx} = -ce^{-x}$

$\frac{dy}{dx} = \frac{-y}{e^{-x}} \cdot e^{-x}$

$\frac{dx}{dy} = y$

$\int dx = \int y\,dy$

$y = \sqrt{x + c}$

$x^2 + y^2 = r^2$

$2x + 2y\frac{dy}{dx} = 0$

$y\frac{-dx}{dy} = -x$

$\int \frac{1}{x} dx = \int \frac{1}{y} dy$

$\ln x + \ln c = \ln y$

$y = xc$

$$y = cx^2 \quad c = \frac{y}{x^2}$$

$$y^2 = cx^3 \quad c = \frac{y^2}{x^3}$$

$$\frac{dy}{dx} = c2x$$

$$2yy' = 3cx^2$$

$$2yy' = \frac{3y^2}{x}$$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$y' = \frac{3y^2}{2x}$$

$$\frac{-dx}{dy} = \frac{2y}{x}$$

$$\frac{-dx}{dy} = \frac{3y}{2x}$$

$$-\int x\,dx = \int 2y\,dy$$

$$-\int 2x\,dx = \int 3y\,dy$$

$$\frac{-x^2}{2} = y^2 + c^2$$

$$-x^2 = 3y^2 + c$$

$$\frac{-x^2}{2} - c^2 = y^2$$

Clairaut

$p = \frac{dy}{dx}$ $\quad y = xp + f(p)$

$y = xy' + 2y'^2$

$y' = y' + xy'' + 4y'y''$

$p = p + xp' + 4pp'$

$p'(x + 4p) = 0$

$\frac{dy}{dx} = 0 \quad x = c$ general solution

$x + 4p = 0$

$p = \frac{-x}{4}$ write as to original

$y = x \cdot \frac{-x}{4} + 2\left(\frac{-x}{4}\right)^2$

$y = \frac{-x^2}{4} + \frac{x^2}{8}$

$y = \frac{-x^2}{8}$

$y = xp + \frac{1}{p}$

$p = p + xp' - \frac{1}{p^2}p'$

$p'\left(x - \frac{1}{p^2}\right) = 0$

$x = c$ general

$x - \frac{1}{p^2} = 0$

$p^2 = \frac{1}{x} \quad p = \frac{1}{\sqrt{x}}$

$y = x \cdot \frac{1}{\sqrt{x}} + \sqrt{x}$

$y = \frac{2x}{\sqrt{x}} \quad y = 2\sqrt{x}$

$$y = xp + p - p^2$$

$$p = p + xp' + p' - 2pp'$$

$$p'(x - 2p + 1) = 0$$

$$x = c$$

$$2p = x + 1$$

$$p = \frac{x+1}{2}$$

$$y = \frac{x^2 + x}{2} + \frac{x+1}{2} - \left(\frac{x^2}{4} + x + \frac{1}{4}\right)$$

$$y = xp + p^4$$

$$p = p + xp' + 4p^3 p'$$

$$p'(x + 4p^3) = 0$$

$$x = c$$

$$p^3 = \frac{-x}{4}$$

$$p = \sqrt[3]{\frac{-x}{4}}$$

$$y = x\sqrt[3]{\frac{-x}{4}} + \left(\sqrt[3]{\frac{-x}{4}}\right)^4$$

$$y = xp + \ln p$$

$$p = p + xp' + \frac{p'}{p}$$

$$p'\left(x + \frac{1}{p}\right) = 0$$

$$x = c$$

$$x + \frac{1}{p} = 0$$

$$p = \frac{-1}{x}$$

$$y = x \cdot \frac{-1}{x} + \ln\left(\frac{-1}{x}\right)$$

$$y + 1 = \ln\left(\frac{-1}{x}\right)$$

$$e^{y+1} = \frac{-1}{x}$$

$$y + 1 = \ln\left(\frac{-1}{x}\right)$$

$$e^{y+1} = e^{\ln\left(\frac{-1}{x}\right)}$$

$$y = xp - 2\sqrt{p}$$

$$p = p + xp' - p^{-\frac{1}{2}} \cdot p'$$

$$p'\left(x - p^{-\frac{1}{2}}\right) = 0$$

$$x = c$$

$$x - p^{-\frac{1}{2}} = 0$$

$$p = \frac{1}{x^2}$$

$$y = \frac{1}{x} \cdot \frac{2}{x} = \frac{-1}{x}$$

$$y = xy' + \sin(y'+1)$$

$$p = p + xp' + p'\cos(p+1)$$

$$p'\left(x + \cos(p+1)\right) = 0$$

$$x = c$$

$$x + \cos(p+1) = 0$$

$$\cos(p+1) = -x$$

$$p + 1 = \arccos(-x)$$

$$p = \arccos(-x) - 1$$

$$y = (\arccos(-x) - 1) \cdot x + \sin(\arccos(-x))$$